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# MECHANICAL CONSTITUTIVE MODELS FOR ENGINEERING MATERIALS

by


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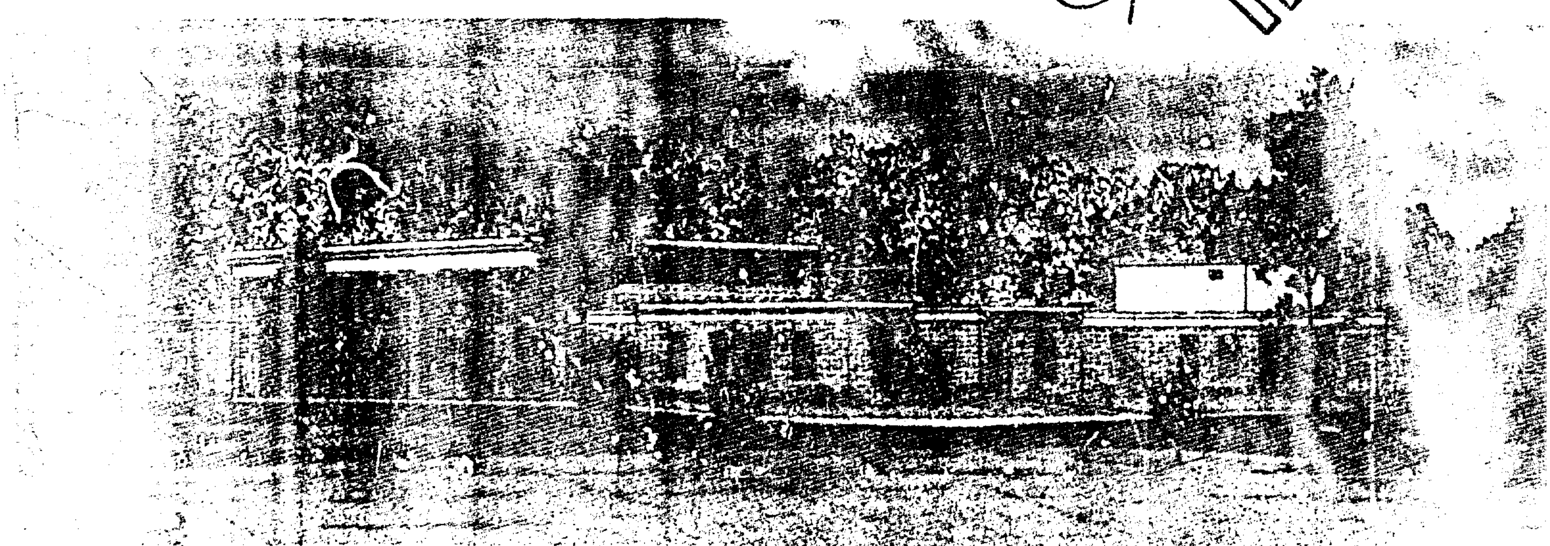


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Some of the basic mathematical tools and physical concepts necessary for development of mechanical constitutive relationships are reviewed and presented in orthogonal Cartesian coordinate system, using both indicial and matrix notations. Following this review, various classes of isotropic constitutive relationships are discussed and examined. The analyses are restricted to deformation for which displacement gradients are small. The
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constitutive relationships are classified and presented in the following categories:

a. Constitutive equations of elastic materials,

b. Incremental constitutive equations,

c. Constitutive equations of simple viscoelastic materials, and

d. Constitutive equations of plastic materials.

## PREFACE

This investigation was conducted by the U. S. Army Engineer Waterways Experiment Station (WES) under Department of the Army Project 4A161101A91D, In-House Laboratory Independent Research Program, sponsored by the Assistant Secretary of the Army (R&D).

The investigation was conducted by Dr. B. Rohani during the calendar years 1975 and 1976 under the general direction of Messrs. J. P. Sale, Chief, Soils and Pavements Laboratory, and Dr. J. G. Jackson, Jr., Chief, Soil Dynamics Division. The report was written by Dr. Rohani.

Directors of WES during the investigation and the preparation of this report were COL G. H. Hilt, CE, and COL J. L. Cannon, CE. Technical Director was Mr. F. R. Brown.

CONTENTS

# MECHANICAL CONSTITUTIVE MODELS FOR ENGINEERING MATERIALS

## PART I: INTRODUCTION

### Background

1. Development of mechanical constitutive models (defined as load-deformation or stress-strain relationships) for engineering materials has received considerable attention in recent years, particularly in the field of geotechnical engineering. The primary reason for such efforts is the fact that with the advent of high-speed electronic computers and the development of new methods of numerical analysis, a variety of complex engineering problems can be solved provided realistic constitutive relationships for the materials of interest are available. Stress-strain relationships for a number of materials, such as soil, rock, and concrete, are often nonlinear even when the magnitudes of the strains involved are small. This type of nonlinear behavior, referred to as physical nonlinearity, has been the subject of investigation at the U. S. Army Engineer Waterways Experiment Station (WES) since early 1960; special emphasis has been placed on modeling the mechanical behavior of earth materials. During the fall of 1971, an elementary course on mechanical constitutive relationships was offered at the Vicksburg Graduate Center, WES, and a series of lecture notes was prepared for use by the students taking this course. The purpose of the lecture notes was to acquaint the students with some of the basic physical concepts and mathematical tools available for developing constitutive relationships. The lecture notes were purposely kept to an elementary level, and were prepared with the formulation of constitutive relations for earth materials in mind.

### Objective

2. The objective of this report is to document the lecture notes

in a format that can be used for engineering training throughout the Corps of Engineers, U. S. Army, or as materials for self-study and reference pruposes.

## Scope

3. Some of the basic mathematical tools necessary for the development of constitutive relationships are presented in Part II. Included in Part II are: a brief discussion of indicial notation, matrix algebra, development of basic equations related to eigenvalue problem, the Cayley-Hamilton theorem, and Cartesian tensors (with emphasis on second-order tensors). A number of numerical examples are included in this part of the report in order to help the reader to better understand the subject matter. Part III includes a summary of appropriate equations from continuum mechanics required for this elementary presentation of the subject of constitutive relationships. Constitutive equations of elastic materials are developed in Part IV. The so-called incremental constitutive equations are discussed in Part V. Constitutive equations of simple viscoelastic materials are discussed in Part VI. Constitutive equations of plasticity are contained in Part VII.

## PART II: MATHEMATICAL PRELIMINARIES

4. Some of the basic mathematical tools necessary for treatment and understanding of the physical concepts to be presented in the ensuing parts of this report are developed in this part. The development is kept to an elementary level and is confined to orthogonal Cartesian coordinate system. In order to establish a common basis of terminology and notation, both indicial and matrix notations are briefly discussed. However, indicial notation is used for most of the presentations throughout this report in order to keep the number of equations to a minimum.

### Indicial Notation

5. The development of indicial notation is based on a number of agreements motivated by miniaturization of a large system of equations or variables. For example, if three variables are denoted by $X_1$, $X_2$, and $X_3$, we can simply denote them by $X_i$, where the subscript i is called an index and we agree that it takes on values 1, 2, and 3 (three-dimensional geometry). Similarly, the system of equations $A_1 = X_1 + Y_1$, $A_2 = X_2 + Y_2$, and $A_3 = X_3 + Y_3$ can be expressed as $A_i = X_i + Y_i$. An index which is not repeated in any single term is called a free index. Thus, the index i in $X_i$ and $A_i = X_i + Y_i$ is a free index. Furthermore, a free index must appear in every term of an expression. Systems which depend on one free index, such as $X_i$ and $A_i$, are called systems of first order. The terms $X_1$, $X_2$, and $X_3$ are called the components or elements of the system. A first-order system, therefore, has three components. Systems which depend on two free indices, such as $A_{ij}$, are called systems of second order. Since the indices take on values 1, 2, and 3, a second-order system has nine components. Similarly, we can define systems of third order which depend on three free indices and have twenty-seven components, e.g., $A_{ijk}$. In this report, however, we will be dealing mainly with first- and second-order systems.

6. If an index appears twice in a term it is called a dummy index. For example, the index $i$ in $A_{ii}$ is a dummy index. By agreement, a dummy index implies that the term is to be summed with respect to this index over the range of the index. Thus $A_{ii} = A_{11} + A_{22} + A_{33}$, $X_iY_i = X_1Y_1 + X_2Y_2 + X_3Y_3$, and $S_{ij} = C_{mm}E_{ij} = (C_{11} + C_{22} + C_{33})E_{ij}$. It is noted that the indices $i$ and $j$ in the last expression are free indices. The particular letter used for the dummy index in an operation is immaterial; thus, $A_{ii} = A_{pp} = A_{mm}$, $X_iY_i = X_pY_p = X_mY_m$, and $S_{ij} = C_{mm}E_{ij} = C_{kk}E_{ij}$. This characteristic of dummy indices is very useful for manipulating several expressions that have common indices. For example, consider the following expressions

$$A_m = B_rC_{mr} \tag{1}$$

$$B_r = D_{mr}E_m \tag{2}$$

In the first expression the index $m$ is free and the index $r$ is a dummy. In the second expression the index $r$ is free and the index $m$ is a dummy. The index $r$ in Equations 1 and 2 is called a connecting index. If we substitute the second expression into the first expression and use the same letters for indices, we obtain

$$A_m = D_{mr}E_mC_{mr} \tag{3}$$

Equation 3 is meaningless since it is not consistent with the rules (agreements) of indicial notation; the index $m$ appears three times on the right-hand side of this expression. To obtain the correct expression we must first overhaul the dummy index $m$ in Equation 2. Using the useful characteristic that the particular letter used for a dummy index is immaterial, we can write

$$B_r = D_{pr}E_p \tag{4}$$

Substituting Equation 4 into Equation 1 we obtain

$$A_m = D_{pr} E_p C_{mr} \tag{5}$$

Equation 5 is notationally correct; there is no question as to which index is the free index. Expanding the dummy indices p and r over their range, Equation 5 takes the following form

$$\begin{aligned} A_m &= D_{p1}E_pC_{m1} + D_{p2}E_pC_{m2} + D_{p3}E_pC_{m3} \\ &= D_{11}E_1C_{m1} + D_{12}E_1C_{m2} + D_{13}E_1C_{m3} \\ &\quad + D_{21}E_2C_{m1} + D_{22}E_2C_{m2} + D_{23}E_2C_{m3} \\ &\quad + D_{31}E_3C_{m1} + D_{32}E_3C_{m2} + D_{33}E_3C_{m3} \end{aligned} \tag{6}$$

Equation 6 (or Equation 5) has three components. The first component, for example, becomes

$$\begin{aligned} A_1 &= D_{11}E_1C_{11} + D_{12}E_1C_{12} + D_{13}E_1C_{13} \\ &\quad + D_{21}E_2C_{11} + D_{22}E_2C_{12} + D_{23}E_2C_{13} \\ &\quad + D_{31}E_3C_{11} + D_{32}E_3C_{12} + D_{33}E_3C_{13} \end{aligned} \tag{7}$$

which is quite long in comparison with the compacted indicial form.

7. Another agreement in establishing indicial notation is the use of commas in the subscripts to represent partial derivatives. Thus, we agree that

$$\frac{\partial F}{\partial X_i} = F_{,i} \tag{8a}$$

$$\frac{\partial U_i}{\partial X_j} = U_{i,j} \tag{8b}$$

Similarly,

$$E_{nk} = \frac{\partial U_m}{\partial X_n}\frac{\partial U_m}{\partial X_k} = U_{m,n}U_{m,k} \qquad (8c)$$

In Equation 8c, m is a dummy index and n and k are free indices. Expanding the dummy index m , Equation 8c takes the following form

$$E_{nk} = U_{1,n}U_{1,k} + U_{2,n}U_{2,k} + U_{3,n}U_{3,k} \qquad (9)$$

Equation 9 (or Equation 8c) has nine components. For example, the $E_{13}$ component becomes

$$E_{13} = U_{1,1}U_{1,3} + U_{2,1}U_{2,3} + U_{3,1}U_{3,3} \qquad (10)$$

8. In indicial notation the condition of symmetry of a second-order system is denoted by

$$B_{ij} = B_{ji} \qquad (11)$$

The condition of skew-symmetry is denoted by

$$C_{ij} = -C_{ji} \qquad (12)$$

Equation 11 results in conditions

$$\left.\begin{aligned} B_{12} &= B_{21} \\ B_{23} &= B_{32} \\ B_{31} &= B_{13} \end{aligned}\right\} \text{conditions of symmetry} \qquad (13)$$

whereas Equation 12 indicates that

$$\left.\begin{array}{l} C_{11} = C_{22} = C_{33} = 0 \\ C_{12} = -C_{21} \\ C_{23} = -C_{32} \\ C_{31} = -C_{13} \end{array}\right\} \text{conditions of skew-symmetry} \qquad (14)$$

Using the above conditions, an asymmetric (i.e., neither symmetric nor skew-symmetric) second-order system $T_{ij}$ can be expressed as the sum of a symmetrical system $1/2\left(T_{ij} + T_{ji}\right)$ and a skew-symmetrical system $1/2\left(T_{ij} - T_{ji}\right)$, i.e.,

$$T_{ij} = 1/2\left(T_{ij} + T_{ji}\right) + 1/2\left(T_{ij} - T_{ji}\right) \qquad (15)$$

9. In using indicial notations, one often deals with quantities that have no free index. Such quantities are referred to as scalars or zero-order systems. For example, the following quantities are scalars

$$\begin{array}{l} A \\ A_{kk} \\ A_{mn}B_{nm} \\ D_{mn}D_{np}D_{pm} \end{array} \qquad (16)$$

It is noted that all indices in Equation 16 are dummy indices. The expanded form of the last expression in Equation 16, for example, becomes

$$
\begin{aligned}
D_{mn}D_{np}D_{pm} &= D_{1n}D_{np}D_{p1} + D_{2n}D_{np}D_{p2} + D_{3n}D_{np}D_{p3} \\
&= D_{11}D_{1p}D_{p1} + D_{12}D_{2p}D_{p1} + D_{13}D_{3p}D_{p1} \\
&\quad + D_{21}D_{1p}D_{p2} + D_{22}D_{2p}D_{p2} + D_{23}D_{3p}D_{p2} \\
&\quad + D_{31}D_{1p}D_{p3} + D_{32}D_{2p}D_{p3} + D_{33}D_{3p}D_{p3} \\
&= D_{11}D_{11}D_{11} + D_{11}D_{12}D_{21} + D_{11}D_{13}D_{31} \\
&\quad + D_{12}D_{21}D_{11} + D_{12}D_{22}D_{21} + D_{12}D_{23}D_{31} \\
&\quad + D_{13}D_{31}D_{11} + D_{13}D_{32}D_{21} + D_{13}D_{33}D_{31} \\
&\quad + D_{21}D_{11}D_{12} + D_{21}D_{12}D_{22} + D_{21}D_{13}D_{32} \\
&\quad + D_{22}D_{21}D_{12} + D_{22}D_{22}D_{22} + D_{22}D_{23}D_{32} \\
&\quad + D_{23}D_{31}D_{12} + D_{23}D_{32}D_{22} + D_{23}D_{33}D_{32} \\
&\quad + D_{31}D_{11}D_{13} + D_{31}D_{12}D_{23} + D_{31}D_{13}D_{33} \\
&\quad + D_{32}D_{21}D_{13} + D_{32}D_{22}D_{23} + D_{32}D_{23}D_{33} \\
&\quad + D_{33}D_{31}D_{13} + D_{33}D_{32}D_{23} + D_{33}D_{33}D_{33}
\end{aligned}
\tag{17}
$$

The compactness of indicial notation is once again demonstrated by the above expansion.

## Matrix Algebra

10. Another convenient method for representing a large number of equations or quantities is through matrix notation. A matrix is an array of numbers or components of a system. For example, the components of a first-order system $X_i$ can be arranged as

$$\{X\} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \tag{18a}$$

or

$$[X] = [X_1 \; X_2 \; X_3] \tag{18b}$$

Equation 18a represents a 3-by-1 (3 rows and 1 column) column matrix whereas Equation 18b represents a 1-by-3 (1 row and 3 columns) row matrix. Similarly, the components of a second-order system $A_{ij}$ can be arranged as

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{19}$$

Equation 19 represents a 3-by-3 square matrix. We are mainly interested in 3-by-3 matrices in this report. Some useful types of matrices are:

<u>a</u>. Diagonal matrix in which all elements other than those on the diagonal are zero.

$$[B] = \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix} \tag{20a}$$

<u>b</u>. Unit matrix in which all off-diagonal elements are zero and every diagonal term is unity.

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20b}$$

c. Symmetrical matrix in which off-diagonal terms are symmetrical.

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} \tag{20c}$$

d. Skew-symmetrical matrix in which every diagonal term is zero and off-diagonal terms are skew-symmetric.

$$[C] = \begin{bmatrix} 0 & C_{12} & C_{13} \\ -C_{12} & 0 & C_{23} \\ -C_{13} & -C_{23} & 0 \end{bmatrix} \tag{20d}$$

In indicial notation the counterparts of Equations 20c and 20d are given by Equations 11 and 12, respectively. Similarly, in indicial form Equation 20a can be expressed as $B_{ij} = 0$ for $i \neq j$ .

11. The transpose $[A]^*$ of a square matrix $[A]$ is obtained by completely interchanging every row with its corresponding column:

$$[A]^* = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \tag{21}$$

In indicial notation $A^*_{ij} = A_{ji}$ . In view of Equations 11 and 12, in the case of a symmetric matrix $B^*_{ij} = B_{ij}$ , and in the case of a skew-symmetric matrix $C^*_{ij} = -C_{ij}$ .

12. Matrices obey certain prescribed rules of matrix algebra. Addition or subtraction of matrices having the same number of rows and the same number of columns is accomplished by adding or subtracting corresponding elements. For example, consider the following 3-by-3 matrices:

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{22a}$$

$$[b] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{22b}$$

Two 3-by-3 matrices can be obtained by adding or subtracting matrices [a] and [b] ; thus,

$$[c] = [a] + [b] = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix} \tag{23a}$$

$$[d] = [a] - [b] = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\ a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \\ a_{31} - b_{31} & a_{32} - b_{32} & a_{33} - b_{33} \end{bmatrix} \tag{23b}$$

In indicial notation the second-order systems [c] and [d] can be expressed by $c_{ij} = a_{ij} + b_{ij}$ and $d_{ij} = a_{ij} - b_{ij}$ . A matrix can be multiplied by a number k by simply multiplying every element in the matrix by k . Two matrices can be multiplied together if they are comformable, i.e., if the number of columns of the first matrix is equal to the number of rows of the second. A p-by-q matrix and a q-by-s matrix are conformable and can be multiplied together. The result of multiplication is a p-by-s matrix. For example, consider the multiplication of matrices [a] and [b] given in Equation 22:

$$[a][b] = [e] \tag{24}$$

The matrix [e] is a 3-by-3 matrix whose components are obtained from the following rule, expressed in indicial notation, governing matrix multiplication:

$$e_{ij} = a_{ik}b_{kj} \tag{25}$$

(e.g., $e_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$ , $e_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$) . From Equation 25 it should be noted that $[a][b] \neq [b][a]$ .

For further examples of matrix multiplication consider the following:

$$\left.\begin{aligned} [a]^2 &= [m] \\ [a]^3 &= [n] \\ [a][b][a] &= [p] \\ [a]^2[b] &= [q] \\ [a]^2[b]^2 &= [r] \\ [a][b][a]^*[b] &= [s] \end{aligned}\right\} \tag{26}$$

Using the rule governing matrix multiplication (Equation 25) it follows that the components of matrices [m] , [n] , [p] , [q] , [r] , and [s] take the following forms

$$\left.\begin{aligned} m_{ij} &= a_{ik}a_{kj} \\ n_{ij} &= a_{ik}a_{kf}a_{fj} \\ p_{ij} &= a_{ik}b_{kf}a_{fj} \\ q_{ij} &= a_{ik}a_{kf}b_{fj} \\ r_{ij} &= a_{ik}a_{kf}b_{fg}b_{gj} \\ s_{ij} &= a_{ik}b_{kf}a^{*}_{fg}b_{gj} = a_{ik}b_{kf}a_{gf}b_{gj} \end{aligned}\right\} \tag{27}$$

Note that in the last expression in Equation 27 the definition $a^{*}_{fg} = a_{gf}$ is invoked. Furthermore, it should be noted that in Equations 25 and 27 the indices i and j are the only free indices.

13. The sum of the diagonal terms of a square matrix is called the <u>trace</u> of the matrix and is denoted by tr (e.g., trace of $[a] = \mathrm{tr}[a] = a_{11} + a_{22} + a_{33}$) . In indicial form,

$$\mathrm{tr}[a] = a_{ii} \tag{28}$$

Similarly, in view of Equations 25 through 27,

$$\left.\begin{aligned} \mathrm{tr}([a][b]) &= a_{ik}b_{ki} \\ \mathrm{tr}[a]^2 &= a_{ik}a_{ki} \\ \mathrm{tr}[a]^3 &= a_{ik}a_{kf}a_{fi} \\ \mathrm{tr}([a][b][a]) &= a_{ik}b_{kf}a_{fi} \end{aligned}\right\} \tag{29}$$

All indices in Equation 29 are dummy indices, indicating that the trace of a matrix is a scalar. Also, tr([a][b]) = tr([b][a]) even though [a][b] $\neq$ [b][a] . This can be verified by expanding the indicial form $a_{ik}b_{ki} = b_{ik}a_{ki}$ .

14. The <u>determinant</u> of a square matrix is denoted by det[a] , or simply $|a|$ , and is expressed as (for a 3-by-3 square matrix)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{30}$$

It is noted that the determinant of a matrix is also a scalar. In conjunction with the determinant of a matrix we define the <u>minor</u> and <u>cofactor.</u> The minor of an element $a_{ij}$ of the matrix [a] is obtained by deleting the $i^{th}$ row and $j^{th}$ column and forming the determinant of the remaining terms. For example, the minor of $a_{12}$ element is given as

$$\text{minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} \tag{31}$$

The cofactor of an element $a_{ij}$ is the minor of that element with a sign attached to it according to the following criterion

$$\text{cofactor of } a_{ij} = (-1)^{i+j} \text{ minor of } a_{ij} \tag{32}$$

Thus, the cofactor of $a_{12}$ element is given as

$$\begin{aligned} \text{cofactor of } a_{12} &= (-1)^{1+2}(a_{21}a_{33} - a_{23}a_{31}) \\ &= -(a_{21}a_{33} - a_{23}a_{31}) \end{aligned} \tag{33}$$

In view of the definition of minor and cofactor the determinant of

the matrix $[a]$ can be expressed as

$$|a| = a_{11}(\text{cofactor of } a_{11}) + a_{12}(\text{cofactor of } a_{12})$$
$$+ a_{13}(\text{cofactor of } a_{13}) \tag{34}$$

It should be pointed out that Equation 34 is not unique in calculating the determinant of the matrix $[a]$. The same final products will result from expansion on columns or other rows, e.g.,

$$|a| = a_{12}(\text{cofactor of } a_{12}) + a_{22}(\text{cofactor of } a_{22})$$
$$+ a_{32}(\text{cofactor of } a_{32}) \tag{35}$$

15. Finally, we define the <u>inverse</u> $[a]^{-1}$ of a square matrix $[a]$ such that

$$[a]^{-1}[a] = [a][a]^{-1} = [I] \tag{36}$$

The inverse matrix is given by

$$[a]^{-1} = \frac{[A]}{|a|} \tag{37}$$

where the matrix $[A]$, called the <u>adjoint</u> of $[a]$, is determined by replacing the elements of $[a]^*$ by their corresponding cofactors; thus,

$$[A] = \begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} & a_{32}a_{13} - a_{12}a_{33} & a_{12}a_{23} - a_{22}a_{13} \\ a_{31}a_{23} - a_{21}a_{33} & a_{11}a_{33} - a_{31}a_{13} & a_{21}a_{13} - a_{11}a_{23} \\ a_{21}a_{32} - a_{31}a_{22} & a_{31}a_{12} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} \tag{38}$$

From Equation 37 it follows that the inverse exists provided $|a| \neq 0$.

## <u>Solutions of Linear Algebraic Equations</u>

16. Consider a set of linear algebraic equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = k_3 \end{aligned}\right\} \tag{39}$$

In indicial notation Equation 39 can be expressed as

$$a_{ij}x_j = k_i \tag{40}$$

In matrix notation Equation 39 takes the following form

$$[a]\{x\} = \{k\} \tag{41}$$

where $[a]$ is a square matrix of coefficients, $\{x\}$ is a column matrix of unknowns, and $\{k\}$ is a column matrix with known elements. The objective is to solve for the elements of the column matrix $\{x\}$. Premultiplying both sides of Equation 41 by $[a]^{-1}$ results in

$$[a]^{-1}[a]\{x\} = [a]^{-1}\{k\} \tag{42}$$

or, in view of Equation 36,

$$\lceil I \rfloor\{x\} = \{x\} = [a]^{-1}\{k\} \tag{43}$$

From Equation 43 it follows that once the inverse of the coefficient matrix is determined the solution for $\{x\}$ is obtained by performing the indicated matrix multiplication.

## Eigenvalue Problem

17. In a number of engineering problems the following system of algebraic equations is often encountered

$$([a] - \lambda\lceil I \rfloor)\{x\} = \{0\} \tag{44}$$

where the elements of the column matrix $\{x\}$ are the unknowns to be determined, $\lambda$ is a scalar parameter, and $\{0\}$ is a null column matrix (all elements being zero). According to Equations 43 and 37, a nontrivial solution of Equation 44 exists only if the determinant of the coefficient matrix vanishes, i.e.,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0 \tag{45}$$

The expansion of the above determinant yields the following cubic equation in $\lambda$ :

$$\lambda^3 - I_a\lambda^2 + II_a\lambda - III_a = 0 \tag{46}$$

where

$$I_a = tr[a] = a_{nn} \tag{47a}$$

$II_a$ = sum of the minors of the diagonal elements of $[a]$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{47b}$$

$$III_a = |a| \tag{47c}$$

Equation 46 is called the characteristic equation of the matrix $[a]$ .

18. The three roots $\lambda_1$ , $\lambda_2$ , and $\lambda_3$ of the characteristic equation are called the characteristic values or eigenvalues of $[a]$ . For every eigenvalue $\lambda_i$ (assuming that all three roots are distinct) Equation 46 is satisfied and hence Equation 44 has nontrivial solutions:

$$\{x(\lambda_i)\} = \begin{Bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{Bmatrix} \tag{48}$$

Every such solution of $\{x\}$ is called a <u>characteristic vector</u> or <u>eigenvector</u> of $[a]$. The eigenvectors $\{x(\lambda_i)\}$ corresponding to eigenvalues $\lambda_i$ can be grouped together to form a square matrix referred to as a <u>modal column matrix</u>, i.e.,

$$[x] = \begin{bmatrix} \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} & \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} & \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \tag{49}$$

For each eigenvalue and the corresponding eigenvector, Equation 44 can be written as

$$[a]\{x(\lambda_i)\} = \lambda_i\{x(\lambda_i)\} \tag{50}$$

Since $[a]$ is a 3-by-3 matrix, Equation 50 can be expressed in the following form for all of the eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{51}$$

Equation 50 is also satisfied if each eigenvector is multiplied by an arbitrary constant $c_i$, i.e.,

$$[a]c_i\{x(\lambda_i)\} = c_i\lambda_i\{x(\lambda_i)\} \tag{52}$$

Therefore, an eigenvector is indeterminate to the extent that it can be multiplied by an arbitrary constant. Selecting an eigenvector

$$\{\mu(\lambda_i)\} = \begin{Bmatrix} \mu_{1i} \\ \mu_{2i} \\ \mu_{3i} \end{Bmatrix} = c_i \begin{Bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{Bmatrix} \tag{53}$$

appropriate to eigenvalue $\lambda_i$ , the corresponding modal column matrix becomes

$$[\mu] = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix} = \begin{bmatrix} c_1x_{11} & c_2x_{12} & c_3x_{13} \\ c_1x_{21} & c_2x_{22} & c_3x_{23} \\ c_1x_{31} & c_2x_{32} & c_3x_{33} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \tag{54}$$

It is observed from Equation 54 that a modal column matrix is indeterminate to the extent that it can be postmultiplied by a diagonal matrix of arbitrary constants $c_i$ . Now utilizing the modal column matrix $[\mu]$ , Equation 51 can be expressed as

$$[a][\mu] = [\mu]\lceil\Lambda\rfloor \tag{55}$$

where $\lceil\Lambda\rfloor$ is a diagonal matrix with elements $\lambda_1$ , $\lambda_2$ , and $\lambda_3$ . Premultiplying Equation 55 by $[\mu]^{-1}$ we obtain

$$[\mu]^{-1}[a][\mu] = \lceil\Lambda\rfloor \tag{56}$$

Postmultiplying Equation 55 by $[\mu]^{-1}$ gives

$$[a] = [\mu]\lceil\Lambda\rfloor[\mu]^{-1} \tag{57}$$

From Equation 56 it is observed that the modal column matrix $[\mu]$ which is found by grouping the eigenvectors of $[a]$ diagonalizes the

matrix [a] . Furthermore, the elements of the diagonalized matrix are the eigenvalues of [a] . This diagonalization process is an important part of the eigenvalue problem and its significance will be realized when dealing with second-order systems.

19. As an example of an eigenvalue problem, consider the following system of equations

$$(2 - \lambda)x_1 - x_2 + x_3 = 0$$
$$-8x_1 + (3 - \lambda)x_2 + 7x_3 = 0$$
$$-8x_1 - x_2 + (11 - \lambda)x_3 = 0$$

In matrix form the above system of equations is expressed as

$$\left( \begin{bmatrix} 2 & -1 & 1 \\ -8 & 3 & 7 \\ -8 & -1 & 11 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

or (see Equation 44)

$$([a] - \lambda[I])\{x\} = \{0\}$$

The characteristic equation of [a] is given as (See Equations 45 and 46)

$$\begin{vmatrix} 2 - \lambda & -1 & 1 \\ -8 & 3 - \lambda & 7 \\ -8 & -1 & 11 - \lambda \end{vmatrix} = -\lambda^3 + 16\lambda^2 - 68\lambda + 80 = 0$$

where it is noted that $I_a = 16$ , $II_a = 68$ , and $III_a = 80$ . Solution of the characteristic equation yields the following eigenvalues for the matrix [a] :

$$\lambda_1 = 2 \text{ , } \lambda_2 = 4 \text{ , } \lambda_3 = 10$$

For each value of $\lambda$ there exists three homogeneous equations. For $\lambda = \lambda_1 = 2$ we have

$$-x_2 + x_3 = 0$$

$$-8x_1 + x_2 + 7x_3 = 0$$

$$-8x_1 - x_2 + 9x_3 = 0$$

where it is noticed that $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ is a nontrivial solution. The eigenvector corresponding to $\lambda_1$ then becomes (see Equation 53)

$$\{\mu(\lambda_1)\} = c_1 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Similarly, for $\lambda = \lambda_2 = 4$,

$$\{\mu(\lambda_2)\} = c_2 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

and for $\lambda = \lambda_3 = 10$,

$$\{\mu(\lambda_3)\} = c_3 \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

The modal column matrix becomes (see Equation 54)

$$[\mu] = \begin{bmatrix} c_1 & c_2 & 0 \\ c_1 & -c_2 & c_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Using Equation 56 it can be verified that the modal column matrix transforms the matrix $[a]$ into a diagonal matrix with elements $\lambda_1$ , $\lambda_2$ , and $\lambda_3$ , i.e.,

$$\begin{bmatrix} c_1 & c_2 & 0 \\ c_1 & -c_2 & c_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 1 \\ -8 & 3 & 7 \\ -8 & -1 & 11 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & 0 \\ c_1 & -c_2 & c_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

## Cayley-Hamilton Theorem

20. The Cayley-Hamilton theorem plays an important role in expressing higher powers of square matrices. It simply states that a square matrix satisfies its own characteristic equation. The result of the theorem is given here without proof. Let $[a]$ be a 3-by-3 matrix and its characteristic equation be given as (see Equation 46)

$$\lambda^3 - I_a\lambda^2 + II_a\lambda - III_a = 0 \tag{58}$$

If $[a]$ satisfies its characteristic equation it follows that

$$[a]^3 = III_a[I] - II_a[a] + I_a[a]^2 \tag{59}$$

Note that the constant $III_a$ is multiplied by a unit matrix $[I]$ . From Equation 59 it follows that

$$[a]^4 = [a]^3[a] = I_aIII_a[I] + (III_a - I_aII_a)[a] + \left(I_a^2 - II_a\right)[a]^2 \tag{60}$$

Similarly,

$$[a]^5 = [a]^4[a] = \left(III_a I_a^2 - III_a II_a\right)[I] + \left(I_a III_a - I_a^2 II_a + II_a^2\right)[a] + \left(I_a^3 - 2I_a II_a + III_a\right)[a]^2 \qquad (61)$$

It is clear from the examples given in Equations 60 and 61 that using the Cayley-Hamilton theorem (i.e., Equation 59) we can express any power of $[a]$ greater than 3 in terms of $[a]$ and $[a]^2$. Accordingly, a polynomial representation of $[a]$, i.e.,

$$[g] = f([a]) = k_0[I] + k_1[a] + k_2[a]^2 + k_3[a]^3 + \ldots + k_n[a]^n \qquad (62)$$

where $k_0$, $k_1$, ..., $k_n$ are constants, can be expressed as

$$[g] = \eta_0[I] + \eta_1[a] + \eta_2[a]^2 \qquad (63)$$

where the coefficients $\eta_0$, $\eta_1$, and $\eta_2$ are now polynomial functions of $I_a$, $II_a$, and $III_a$.

21. For an illustrative example of the Cayley-Hamilton theorem, consider the following matrix

$$[c] = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$

The characteristic equation of $[c]$ is given as

$$\lambda^3 + 3\lambda^2 - 7\lambda - 17 = 0$$

where it is noted that

$$I_c = -3$$

$$II_c = -7$$

$$III_c = 17$$

We substitute [c] for $\lambda$ in the characteristic equation and multiply the constant term by a unit matrix [I] , i.e.,

$$\begin{bmatrix}1&2&0\\3&-1&-2\\1&0&-3\end{bmatrix}^3 + 3\begin{bmatrix}1&2&0\\3&-1&-2\\1&0&-3\end{bmatrix}^2 - 7\begin{bmatrix}1&2&0\\3&-1&-2\\1&0&-3\end{bmatrix}$$

$$-17\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}3&14&12\\27&-11&-38\\13&-6&-31\end{bmatrix} + \begin{bmatrix}21&0&-12\\-6&21&24\\-6&6&27\end{bmatrix}$$

$$-\begin{bmatrix}7&14&0\\21&-7&-14\\7&0&-21\end{bmatrix} - \begin{bmatrix}17&0&0\\0&17&0\\0&0&17\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

resulting in a null matrix [0] .

## Cartesian Tensors

### Cartesian coordinate

22. Let us consider the orthogonal Cartesian coordinate system $x_k$ (Figure 1) with unit vectors $i_1$ , $i_2$ , and $i_3$ along the $x_1$ , $x_2$ , and $x_3$ axes, respectively.

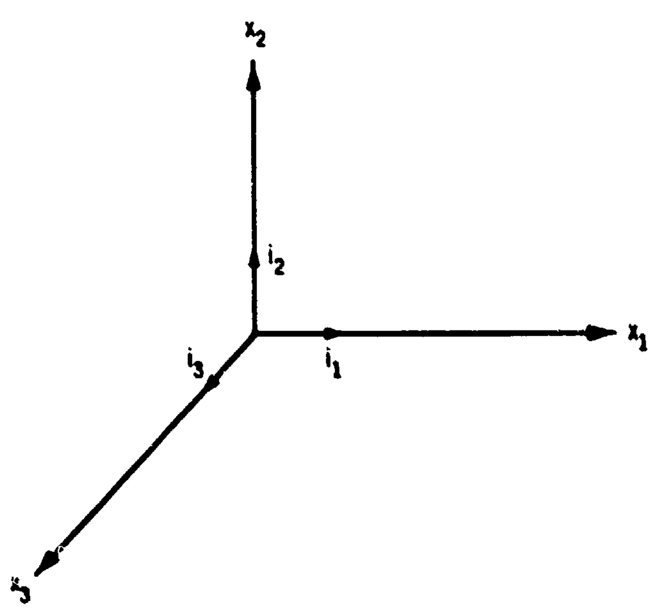



Figure 1. Orthogonal Cartesian coordinate system $x_k$

From elementary vector analysis the dot products of these unit vectors are given as

$$\left.\begin{aligned} i_1 \cdot i_1 = i_2 \cdot i_2 = i_3 \cdot i_3 = 1 \\ i_1 \cdot i_2 = i_1 \cdot i_3 = i_2 \cdot i_3 = 0 \end{aligned}\right\} \tag{64}$$

or, in indicial notation,

$$i_p \cdot i_r = \begin{cases} 0 & p \neq r \\ 1 & p = r \end{cases} \tag{65}$$

This product is denoted by $\delta_{pr}$ and is known as the <u>Kronecker delta</u>; thus,

$$i_p \cdot i_r = \delta_{pr} = \begin{cases} 0 & p \neq r \\ 1 & p = r \end{cases} \tag{66}$$

The counterpart of $\delta_{pr}$ in matrix notation is the unit matrix $[I]$ (see Equation 20b). From Equations 65 and 66 it follows that

$$\left.\begin{aligned} \delta_{pp} = 3 \\ \delta_{pr}\delta_{rp} = 3 \end{aligned}\right\} \tag{67}$$

<u>Transformation matrix</u>

23. The vector $\bar{V}$ with components $(x_1 , x_2 , x_3)$ in the $x_k$ coordinate system (Figure 2) can be expressed in vector form as

$$\bar{V} = x_1 i_1 + x_2 i_2 + x_3 i_3 = x_p i_p \tag{68}$$

If we fix the origin and rotate the axes forming a new coordinate system $x'_k$ , with corresponding unit vectors $i'_k$ (Figure 2), then the vector

$\overline{V}$ with components $(x'_1, x'_2, x'_3)$ in the primed (rotated) system can be denoted by

$$\overline{V} = x'_s i'_s \tag{69}$$

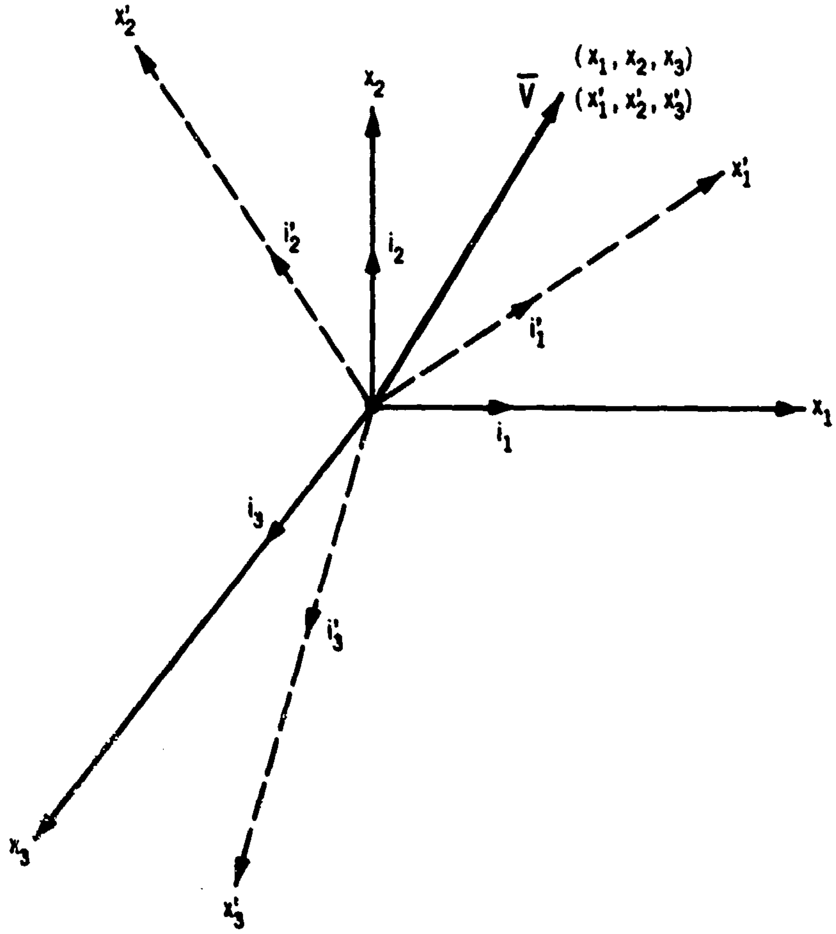



Figure 2. Orthogonal Cartesian coordinate systems $x_k$ and $x'_k$

In view of Equations 68 and 69

$$x_p i_p = x'_s i'_s \tag{70}$$

The dot product of Equation 70 with $i_k$ results in

$$x_p i_p \cdot i_k = x'_s i'_s \cdot i_k \tag{71}$$

Since $i_p \cdot i_k = \delta_{pk}$, Equation 71 reduces to

$$x_k = x'_s i'_s \cdot i_k \tag{72}$$

By the definition of dot product,

$$i'_s \cdot i_k = \cos(x'_s , x_k) \tag{73}$$

where $\cos(x'_s , x_k)$ is the cosine of the angle between the $x'_s$ and $x_k$ axes. We denote $\cos(x'_s , x_k)$ by $a_{sk}$

$$a_{sk} = \cos(x'_s , x_k) \tag{74}$$

In view of Equations 73 and 74, Equation 72 takes the form

$$x_k = a_{sk} x'_s \tag{75}$$

Similarly, the dot product of Equation 70 with $i'_k$ results in the following relation

$$x'_k = a_{ks} x_s \tag{76}$$

Equation 75 relates the components of the primed system (rotated) to the components of the unprimed system. Equation 76 relates the components of the unprimed system to the components of the primed system. The matrix $a_{sk}$ is called the transformation matrix and consists of the following table of direction cosines:

| Table of Direction Cosines | | | |
|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ |
| $x'_1$ | $a_{11}$ | $a_{12}$ | $a_{13}$ |
| $x'_2$ | $a_{21}$ | $a_{22}$ | $a_{23}$ |
| $x'_3$ | $a_{31}$ | $a_{32}$ | $a_{33}$ |

where $a_{11} = \cos(x'_1 , x_1)$ , $a_{21} = \cos(x'_2 , x_1)$ , $a_{12} = \cos(x'_1 , x_2)$, etc.

24. Equations 75 and 76 can now be utilized to establish certain properties of the transformation matrix. Differentiating Equation 75 with respect to $x_i$ yields

$$x_{k,i} = a_{sk} x'_{s,i} \tag{77}$$

Since $x_{k,i} = \delta_{ki}$ , i.e., $x_{1,1} = 1$ , $x_{1,2} = 0$ , etc., Equation 77 reduces to

$$\delta_{ki} = a_{sk} x'_{s,i} \tag{78}$$

From Equation 76, $x'_s = a_{sm} x_m$ , and thus

$$x'_{s,i} = a_{sm} x_{m,i} = a_{sm} \delta_{mi} \tag{79}$$

Substituting Equation 79 into Equation 78 results in

$$\delta_{ki} = a_{sk} a_{sm} \delta_{mi} \tag{80}$$

In view of the definition of $\delta_{mi}$ , Equation 80 reduces to

$$a_{sk} a_{si} = \delta_{ki} \tag{81}$$

Similarly, by differentiating Equation 76 and following the same procedure we get

$$a_{kp}a_{ip} = \delta_{ki} \tag{82}$$

Equations 81 and 82 describe the basic properties of the transformation matrix. Expanding Equation 81 yields

$$\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = \delta_{11} = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{cases} \tag{83a}$$

$$\begin{cases} a_{11}\, a_{12} + a_{21}\, a_{22} + a_{31}a_{32} = \delta_{12} = 0 \\ a_{11}\, a_{13} + a_{21}\, a_{23} + a_{31}\, a_{33} = 0. \\ a_{12}\, a_{13} + a_{22}\, a_{23} + a_{32}\, a_{33} = 0 \end{cases} \tag{83b}$$

Similarly, expanding Equation 82 yields

$$\begin{cases} a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases} \tag{84a}$$

$$\begin{cases} a_{11}\, a_{21} + a_{12}\, a_{22} + a_{13}\, a_{23} = 0 \\ a_{11}\, a_{31} + a_{12}\, a_{32} + a_{13}\, a_{33} = 0 \\ a_{21}\, a_{31} + a_{22}\, a_{32} + a_{23}\, a_{33} = 0 \end{cases} \tag{84b}$$

Equations 83a and 84a indicate that the sum of the squares of the elements of any column or row of the transformation matrix is unity and

called <u>normalization conditions.</u> Equations 83b and 84b indicate that the sum of the products of corresponding elements in any two distinct columns or rows is zero and called <u>orthogonality conditions.</u> Through algebraic manipulations of Equations 83 and 84 it can also be shown that

$$|a|^2 = 1 \tag{85}$$

25. For a numerical example of a transformation matrix consider the following rotation (Figure 3) of the $x_k$ coordinate system:

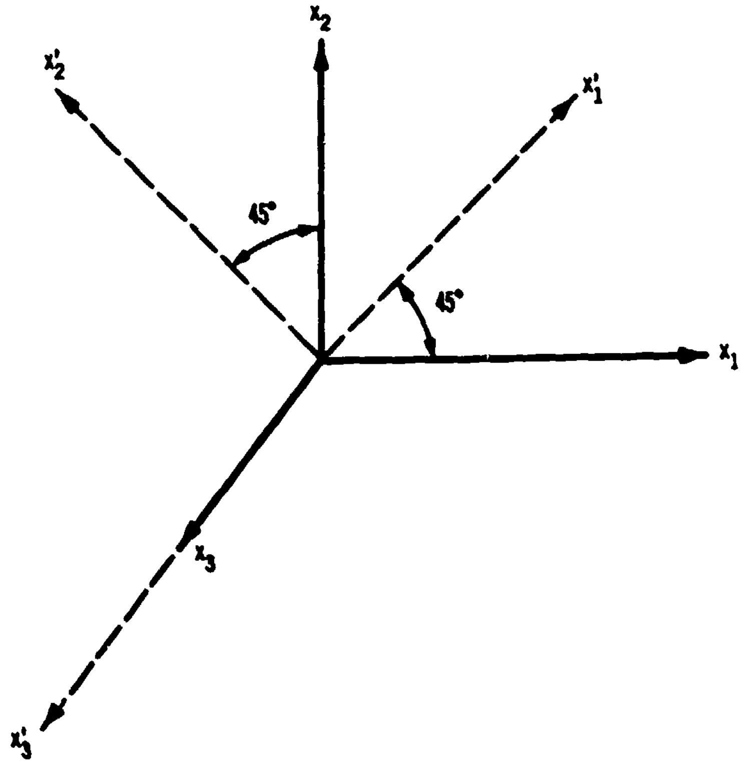



Figure 3. Rigid-body rotation of $x_k$ coordinate system

The transformation matrix $a_{sk}$ associated with this rotation can be constructed easily:

$$a_{sk} = \cos(x_s', x_k) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It can readily be verified that the above matrix satisfies Equations 83 through 85.

First-order tensor

26. If with a coordinate transformation $x'_k = a_{ks}x_s$ (see Equation 76), the three quantities $A_s$ in the unprimed coordinate system transform to three quantities $A'_k$ in the primed reference frame by

$$A'_k = a_{ks}A_s \tag{86}$$

then $A_s$ is a tensor of the first order. We already know that $A_s$ is a vector. Therefore, a vector is a tensor of first order. Within the context of indicial notation, a first-order tensor is a first-order system, i.e., it has one free index. Any quantity whose value does not change with coordinate transformation is called a tensor of order zero or a scalar (see Equation 16). A scalar is, therefore, invariant to rigid-body rotation of the coordinate system. Considering the scalar product of $A'_k$ with itself we obtain

$$A'_kA'_k = a_{ks}A_sa_{kp}A_p \tag{87}$$

Since $a_{ks}a_{kp} = \delta_{sp}$ (see Equation 81), Equation 87 becomes

$$A'_kA'_k = A_sA_p\delta_{sp} \tag{88}$$

In view of the definition of $\delta_{sp}$, Equation 88 reduces to

$$A'_kA'_k = A_sA_s \tag{89}$$

Equation 89 indicates that the sum of the square of the elements (components) of a first-order tensor (vector) is invariant to rigid-body rotation of the coordinate axes. This quantity is the only invariant associated with a first-order tensor. The magnitude or length of the vector $A_s$ is given as $\sqrt{A_sA_s}$ and is, therefore, invariant to rigid-body rotation of the coordinate system.

27. For an example of transformation of a first-order tensor, consider the vector $A_k$ with components

$$A_k = \begin{Bmatrix} 5 \\ 10 \\ 2 \end{Bmatrix}$$

in the $x_k$ coordinate system. The magnitude of the vector is

$$\sqrt{A_k A_k} = \sqrt{(5)^2 + (10)^2 + (2)^2} = \sqrt{129}$$

If the coordinate system undergoes a rigid-body rotation as shown in Figure 3, the components of the vector in the rotated system can be calculated from Equation 86, i.e.,

$$A_1' = a_{11}A_1 + a_{12}A_2 + a_{13}A_3 = \frac{\sqrt{2}}{2}(5) + \frac{\sqrt{2}}{2}(10) = \frac{15\sqrt{2}}{2}$$

$$A_2' = a_{21}A_1 + a_{22}A_2 + a_{23}A_3 = \frac{-\sqrt{2}}{2}(5) + \frac{\sqrt{2}}{2}(10) = \frac{5\sqrt{2}}{2}$$

$$A_3' = a_{31}A_1 + a_{32}A_2 + a_{33}A_3 = 2$$

It is noted that the magnitude of the vector is not affected by the coordinate transformation, i.e.,

$$\sqrt{A_k' A_k'} = \sqrt{\left(\frac{15\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2}\right)^2 + (2)^2} = \sqrt{129}$$

Second-order tensor

28. Consider two first-order tensors $u_i$ and $v_i$ associated with coordinate system $x_i$. Since $u_i$ and $v_i$ are first-order tensors we may write (see Equation 75)

$$u_i = a_{ni} u_n' \tag{90a}$$

$$v_i = a_{mi} v_m' \tag{90b}$$

Combining the vectors $u_i$ and $v_i$ we can construct the second-order system $u_i v_j$, which we may call the array $T_{ij}$, i.e.,

$$u_i v_j = T_{ij} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix} \tag{91}$$

In view of Equation 90 the product $u_i v_j$ can be written as

$$T_{ij} = u_i v_j = a_{ni} u'_n a_{mj} v'_m = a_{ni} a_{mj} u'_n v'_m \tag{92}$$

Equation 92 provides the array of nine-number $T_{ij}$. Denoting the array $u'_n v'_m$ by $T'_{nm}$, Equation 92 becomes

$$T_{ij} = a_{ni} a_{mj} T'_{nm} \tag{93}$$

where $T'_{nm}$ is referred to the primed coordinate system. Similarly, starting from $u'_i = a_{in} u_n$ and $v'_i = a_{im} v_m$, we can derive

$$T'_{ij} = a_{in} a_{jm} T_{nm} \tag{94}$$

Any quantity $T_{nm}$ that transforms according to Equation 94 is called a second-order tensor. Within the context of indicial notation, a second-order tensor is a second-order system, i.e., it has two free indices. Accordingly, the addition, subtraction, and multiplication of second-order tensors are governed by the rules expressed in Equations 23 through 27. In matrix notation the transformation laws (Equations 93 and 94) are expressed as

$$[T] = [a]^* [T'] [a] \tag{95}$$

$$[T'] = [a] [T] [a]^* \tag{96}$$

where $[a]^*$ is the transpose of $[a]$ .

29. The second-order tensor is an extremely important tensor in mechanics and will be used extensively in this report. In particular, we are interested in second-order symmetric tensors such as stress and strain tensors. It was shown in Equation 89 that there is one invariant associated with a first-order tensor (vector). In the case of a second-order symmetric tensor, however, there are three independent quantities that remain constant with respect to coordinate transformation. These independent invariants are

$$I_T = tr[T] = T_{ii} \tag{97}$$

$$\overline{II}_T = tr[T]^2 = T_{ik}T_{ki} \tag{98}$$

$$\overline{III}_T = tr[T]^3 = T_{ik}T_{km}T_{mi} \tag{99}$$

From Equation 94 it follows that

$$tr[T'] = T'_{ii} = a_{in}a_{im}T_{nm} \tag{100}$$

According to the property of the transformation matrix (Equation 81), $a_{in}a_{im} = \delta_{nm}$ , and Equation 100 becomes

$$tr[T'] = \delta_{nm}T_{nm} \tag{101}$$

In view of the definition of $\delta_{nm}$ , Equation 101 reduces to

$$tr[T'] = T_{nn} = tr[T] \tag{102}$$

indicating that $tr[T]$ is an invariant. Similarly, from Equation 94, $T'_{ik} = a_{in}a_{km}T_{nm}$ , $T'_{ki} = a_{kp}a_{is}T_{ps}$ , and

$$\mathrm{tr}[T']^2 = T'_{ik}T'_{ki}$$

$$= a_{in}a_{km}T_{nm}a_{kp}a_{is}T_{ps}$$

$$= a_{in}a_{is}T_{nm}a_{km}a_{kp}T_{ps} \qquad (103)$$

Again using the property of the transformation matrix (Equation 81), $a_{in}a_{is} = \delta_{ns}$ , $a_{km}a_{kp} = \delta_{mp}$ , and Equation 103 becomes

$$\mathrm{tr}[T']^2 = \delta_{ns}T_{nm}\delta_{mp}T_{ps} \qquad (104)$$

In view of the definitions of $\delta_{ns}$ and $\delta_{mp}$ , Equation 104 reduces to

$$\mathrm{tr}[T']^2 = T_{sm}T_{ms} = \mathrm{tr}[T]^2 \qquad (105)$$

indicating that $\mathrm{tr}[T]^2$ is an invariant. Using the same procedure it can be shown that

$$\mathrm{tr}[T']^3 = \mathrm{tr}[T]^3 \qquad (106)$$

indicating that $\mathrm{tr}[T]^3$ is also an invariant.

30. The three invariants of the second-order tensor ($I_T$ , $\overline{II}_T$ , $\overline{III}_T$) can be related to the coefficients in the characteristic equation of the tensor (Equation 47). By algebraic manipulation it can be shown that

$$\overline{II}_T = I_T^2 - 2II_T \qquad (107a)$$

$$II_T = \frac{1}{2}\left(I_T^2 - \overline{II}_T\right) \qquad (107b)$$

Now, using the Cayley-Hamilton theorem (Equation 59) in indicial form, i.e.,

$$T_{ik}T_{km}T_{mj} = III_T\delta_{ij} - II_T T_{ij} + I_T T_{in}T_{nj} \qquad (108)$$

and taking the trace of the tensor (putting $i = j$), we obtain

$$T_{ik}T_{km}T_{mi} = 3III_T - II_T T_{ii} + I_T T_{in} T_{ni} \qquad (109)$$

In view of Equations 97, 98, 99, and 107, Equation 109 results in

$$\overline{III}_T = 3III_T - 3II_T I_T + I_T^3 \qquad (110a)$$

$$III_T = \frac{1}{3}\,\overline{III}_T - \frac{1}{2}\,\overline{II}_T I_T + \frac{1}{6}\,I_T^3 \qquad (110b)$$

Equations 107 and 110 indicate that the coefficients in the characteristic equation of the tensor are also invariant.

31. For an illustrative example of transformation of second-order tensors, consider the following tensor associated with an $x_k$ coordinate system:

$$T_{ij} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix}$$

From Equations 97, 98, and 99 we have

$$I_T = tr[T] = 18$$

$$\overline{II}_T = tr[T]^2 = 126$$

$$\overline{III}_T = tr[T]^3 = 966$$

Also, from Equation 47,

$$II_T = 99$$

$$III_T = 160$$

where it is noted that Equations 107 and 110 are satisfied. If the coordinate system $x_k$ undergoes a rigid-body rotation, such as the one

shown in Figure 3, the components of the tensor in the $x'_k$ (rotated) system can be calculated from the transformation law of second-order tensors (Equation 94). The transformation matrix associated with the coordinate rotation in Figure 3 is given as

$$a_{sk} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From Equation 94 it follows that

$$\begin{aligned} T'_{ij} &= a_{in}a_{j1}T_{n1} + a_{in}a_{j2}T_{n2} + a_{in}a_{j3}T_{n3} \\ &= a_{i1}a_{j1}T_{11} + a_{i2}a_{j1}T_{21} + a_{i3}a_{j1}T_{31} \\ &\quad + a_{i1}a_{j2}T_{12} + a_{i2}a_{j2}T_{22} + a_{i3}a_{j2}T_{32} \\ &\quad + a_{i1}a_{j3}T_{13} + a_{i2}a_{j3}T_{23} + a_{i3}a_{j3}T_{33} \end{aligned}$$

Substituting for the components of $T_{ij}$ and $a_{sk}$, we obtain

$$T'_{ij} = \begin{bmatrix} 6 & 1 & \sqrt{2} \\ 1 & 4 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 8 \end{bmatrix}$$

Now, utilizing Equations 97, 98, 99, and 47 we obtain

$$I_{T'} = 18 = I_T$$

$$\overline{II}_{T'} = 126 = \overline{II}_T$$

$$\overline{III}_{T'} = 966 = \overline{III}_T$$

$$II_{T'} = 99 = II_T$$

$$III_{T'} = 160 = III_T$$

indicating the invariant nature of these quantities.

32. We now proceed to establish some useful relationships for second-order tensors. Consider a second-order symmetric tensor whose elements in the $x_k$ coordinate system are given as $T_{ij} = T_{ji}$ . Using the transformation law of second-order tensors, we seek a transformation matrix that will transform $T_{ij}$ into a diagonal form $T'_{ij}$ (i.e., $T'_{ij} = 0$ for $i \neq j$) associated with an $x'_k$ coordinate system. The axes $x'_k$ are called the principal axes (or principal directions) of the tensor and the elements of $T'_{ij}$ are called the principal values of the tensor. A diagonalization process was previously demonstrated for 3-by-3 matrices in conjunction with the eigenvalue problem. It was shown that the modal column matrix, which is found by grouping the eigenvectors of a square matrix, diagonalizes the matrix as indicated by Equation 56. Furthermore, it was shown that an eigenvector is indeterminate to the extent that it can be multiplied by an arbitrary constant. If the arbitrary constant is chosen to be the inverse of the length or magnitude of the eigenvector, then the eigenvector is said to be normalized. The modal column matrix of normalized eigenvectors is called a normalized modal column matrix. Denoting the normalized modal column matrix by $[\mu]$ , the diagonalization relation (Equation 56) for the matrix $[T]$ can be written as

$$[\Lambda] = [\mu]^{-1}[T][\mu] \qquad (111)$$

In view of the properties of a transformation matrix, the transformation law of a second-order tensor (Equation 96) can be written as

$$[T'] = \left[[a]^*\right]^{-1} [T][a]^* \qquad (112)$$

Comparison of Equation 112 with Equation 111 indicates that the transpose of the normalized modal column matrix is the transformation matrix

which transforms $T_{ij}$ into a diagonal form. Furthermore, the elements of the diagonalized matrix are the eigenvalues of $T_{ij}$. In the case of second-order symmetric tensors, the eigenvalues (principal values) are always real. It should be noted that the normalization of eigenvectors is necessary in order to conform with the normalization conditions of the transformation matrix (Equations 83a and 84a).

33. For a numerical example of diagonalization of a second-order symmetric tensor consider the tensor $T_{ij}$ whose elements in the $x_k$ coordinate system are given as

$$T_{ij} = \begin{bmatrix} -2 & 2 & 10 \\ 2 & -11 & 8 \\ 10 & 8 & -5 \end{bmatrix}$$

The characteristic equation of $T_{ij}$ is given as

$$\begin{vmatrix} -2-\lambda & 2 & 10 \\ 2 & -11-\lambda & 8 \\ 10 & 8 & -5-\lambda \end{vmatrix} = (\lambda - 9)(\lambda + 9)(\lambda + 18) = 0$$

The eigenvalues of $T_{ij}$ are, therefore,

$$\lambda_1 = 9 \ ; \quad \lambda_2 = -9 \ ; \quad \lambda_3 = -18$$

Next, we determine the normalized eigenvectors for $T_{ij}$. For $\lambda = \lambda_1$ we can write down (see Equation 44)

$$-11x_1 + 2x_2 + 10x_3 = 0$$
$$2x_1 - 20x_2 + 8x_3 = 0$$
$$10x_1 + 8x_2 - 14x_3 = 0$$

Solving the above system of equations and considering the normalization

condition of the eigenvector (i.e., $x_1^2 + x_2^2 + x_3^2 = 1$), the normalized eigenvector corresponding to $\lambda_1$ becomes

$$\{\mu(\lambda_1)\} = \begin{Bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{Bmatrix}$$

Similarly, for $\lambda = \lambda_2$ and $\lambda = \lambda_3$, we obtain

$$\{\mu(\lambda_2)\} = \begin{Bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{Bmatrix}$$

$$\{\mu(\lambda_3)\} = \begin{Bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{Bmatrix}$$

The normalized modal column matrix then becomes

$$[\mu] = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

and

$$[\mu]^* = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

As was stated previously, the transformation matrix which transforms $T_{ij}$ into a diagonal form is the transpose of the normalized modal column matrix. This can be verified by using $[\mu]^*$ as the transformation matrix $[a]$ in Equation 96, i.e.,

$$[T'] = [a][T][a]^* = [\mu]^*[T][\mu]$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -2 & 2 & 10 \\ 2 & -11 & 8 \\ 10 & 8 & -5 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Performing the above matrix operation we obtain

$$[T'] = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -18 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

34. Consider three second-order symmetric tensors $A_{ij}$, $B_{mn}$, and $C_{rs}$. Using the Cayley-Hamilton theorem it was shown previously that a polynomial representation relating the components of two tensors takes the form given in Equation 63. In indicial rotation Equation 63 is expressed in the following form

$$A_{ij} = f_{ij}(B_{mn}) = \eta_0\delta_{ij} + \eta_1 B_{ij} + \eta_2 B_{ik}B_{kj} \tag{113}$$

The counterpart of Equation 113 expressing the components of one tensor in terms of the components of two other tensors was derived by Rivlin and Ericksen.[1] The Rivlin-Ericksen equation given here without proof has the following form

$$\begin{aligned} A_{ij} = f_{ij}(B_{mn}, C_{rs}) &= \eta_0\delta_{ij} + \eta_1 B_{ij} + \eta_2 B_{ik}B_{kj} \\ &+ \eta_3 C_{ij} + \eta_4 C_{ik}C_{kj} + \eta_5(B_{ik}C_{kj} + C_{ik}B_{kj}) \\ &+ \eta_6(B_{ik}B_{kp}C_{pj} + C_{ik}B_{kp}B_{pj}) \\ &+ \eta_7(B_{ik}C_{kp}C_{pj} + C_{ik}C_{kp}B_{pj}) \\ &+ \eta_8(B_{ik}B_{kp}C_{pt}C_{tj} + C_{ik}C_{kp}B_{pt}B_{tj}) \end{aligned} \tag{114}$$

where the coefficients $\eta_0, \ldots, \eta_8$ are polynomial functions of the invariants of $B_{mn}$ and $C_{rs}$ and the following joint invariants

$$\left.\begin{aligned} \Pi_1 &= B_{ab}C_{ba} \\ \Pi_2 &= B_{ab}C_{bc}C_{ca} \\ \Pi_3 &= B_{ab}B_{bc}C_{ca} \\ \Pi_4 &= B_{ab}B_{bc}C_{cd}C_{da} \end{aligned}\right\} \tag{115}$$

It is noted that when dependence on $C_{rs}$ disappears, Equation 114 reduces to Equation 113. Equations 113 and 114 are the bases for most of the presentations in the ensuing parts of this report.

## PART III: SUMMARY OF BASIC CONCEPTS FROM CONTINUUM MECHANICS

### Stress Tensor

35. In Cartesian coordinate system $x_i$, we define the stress tensor $\sigma_{ij}$ at a point as

$$\sigma_{ij} = \lim_{A_i \to 0} \frac{F_j}{A_i} \tag{116}$$

where $F_j$ is force in the coordinate direction $j$ and $A_i$ is the area normal to $i^{th}$ axis on which the force $F_j$ acts. Figure 4 depicts the

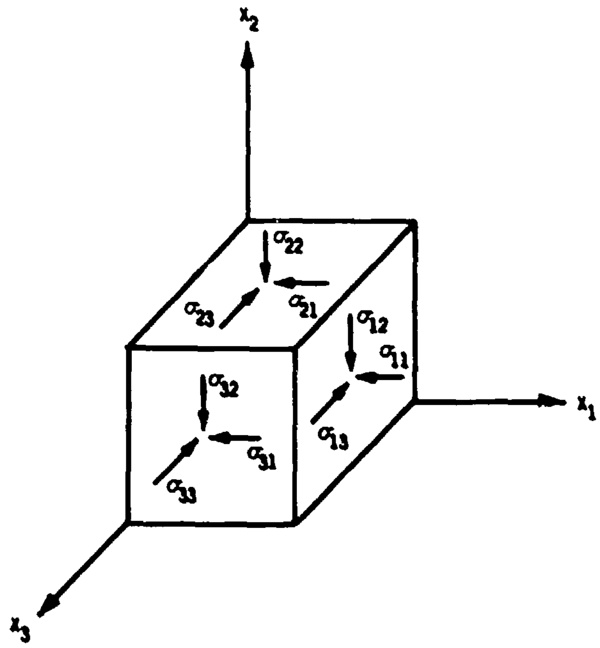



Figure 4. Stress components

positive directions of the components of the stress tensor. In the absence of distributed body or surface couples the stress tensor is symmetrical, i.e., $\sigma_{ij} = \sigma_{ji}$. Accordingly, the state of stress at a point can be described by six independent stress components.

### Invariants of stress tensor

36. Stress tensor is a second-order symmetric tensor and it obeys

the transformation law given in Equation 94, i.e.,

$$\sigma'_{ij} = a_{in} a_{jm} \sigma_{nm} \tag{117}$$

where $\sigma'_{ij}$ is referred to $x'_i$ (rotated) coordinate system. As was shown in Part II, a second-order tensor has three independent invariants (Equations 97, 98, and 99). In the case of stress tensor we define these invariants as

$$J_1 = I_\sigma = \sigma_{nn} \tag{118}$$

$$\overline{J}_2 = \frac{1}{2}\,\overline{II}_\sigma = \frac{1}{2}\,\sigma_{ik}\sigma_{ki} \tag{119}$$

$$\overline{J}_3 = \frac{1}{3}\,\overline{III}_\sigma = \frac{1}{3}\,\sigma_{ik}\sigma_{km}\sigma_{mi} \tag{120}$$

Stress deviation tensor

37. Stress tensor can be expressed as the sum of two second-order symmetric tensors in the following manner

$$\sigma_{ij} = S_{ij} + \frac{1}{3}\,\sigma_{nn}\delta_{ij} \tag{121}$$

where the tensor

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\,\sigma_{nn}\delta_{ij} \tag{122}$$

is referred to as the stress deviation tensor and $\sigma_{nn}\delta_{ij}/3$ is called the spherical stress tensor. An important property of the stress deviation tensor is that its trace is equal to zero, i.e.,

$$S_{ii} = \sigma_{ii} - \sigma_{nn} = 0 \tag{123}$$

The stress deviation tensor, therefore, has only two independent invariants. We denote these invariants as

$$\bar{J}_2' = \frac{1}{2}\,\overline{II}_S = \frac{1}{2}\,S_{ik}S_{ki} \tag{124}$$

$$\bar{J}_3' = \frac{1}{3}\,\overline{III}_S = \frac{1}{3}\,S_{ik}S_{km}S_{mi} \tag{125}$$

The invariants of stress and stress deviation tensors can be related by using Equation 122. In view of Equations 122 and 118,

$$\begin{aligned}
\bar{J}_2' &= \frac{1}{2}\,S_{ik}S_{ki} \\
&= \frac{1}{2}\left(\sigma_{ik} - \frac{1}{3}\,J_1\delta_{ik}\right)\left(\sigma_{ki} - \frac{1}{3}\,J_1\delta_{ki}\right) \\
&= \frac{1}{2}\left(\sigma_{ik}\sigma_{ki} - \frac{2}{3}\,J_1\sigma_{ik}\delta_{ki} + \frac{J_1^2}{9}\,\delta_{ik}\delta_{ki}\right)
\end{aligned} \tag{126}$$

Since $\delta_{ik}\delta_{ki} = 3$ , $\sigma_{ik}\delta_{ki} = J_1$ , and $\sigma_{ik}\sigma_{ki} = 2\bar{J}_2$ , Equation 126 becomes

$$\bar{J}_2' = \bar{J}_2 - \frac{1}{6}\,J_1^2 \tag{127}$$

Similarly, it can be shown that

$$\bar{J}_3' = \bar{J}_3 - \frac{2}{3}\,J_1\bar{J}_2 + \frac{2}{27}\,J_1^3 \tag{128}$$

Principal stresses

38. The three principal values of stress tensor are referred to as principal stresses and are denoted by (using the principal directions as reference axes)

$$[\sigma] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{129}$$

It should be pointed out that the ordering of the principal stresses in Equation 129 does not imp'" that the numerical value of $\sigma_1$ is greater than $\sigma_2$ or $\sigma_3$ . As discussed in Part II, the three principal values are the roots of the characteristic equation of the tensor

$$\lambda^3 - I_\sigma \lambda^2 + II_\sigma \lambda - III_\sigma = 0 \tag{130}$$

where

$$I_\sigma = J_1 = \sigma_{nn} \tag{131a}$$

$$II_\sigma = \begin{vmatrix} \sigma_{22} \sigma_{23} \\ \sigma_{32} \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} \sigma_{13} \\ \sigma_{31} \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} \sigma_{12} \\ \sigma_{21} \sigma_{22} \end{vmatrix} \tag{131b}$$

$$III_\sigma = |\sigma| \tag{131c}$$

The two coefficients $II_\sigma$ and $III_\sigma$ are usually denoted by $J_2$ and $J_3$, respectively, and can be related to $J_1$, $\bar{J}_2$, and $\bar{J}_3$ by using Equations 107, 110, 118, 119, and 120.

$$J_2 = II_\sigma = \frac{1}{2}\left(J_1^2 - 2\bar{J}_2\right) \tag{132}$$

$$J_3 = III_\sigma = \bar{J}_3 - \bar{J}_2 J_1 + \frac{1}{6} J_1^3 \tag{133}$$

The invariants of stress deviation tensor can also be expressed in terms of $J_1$, $\bar{J}_2$, and $J_3$. In view of Equations 127, 128, 132, and 133, we obtain

$$\bar{J}'_2 = \frac{1}{3} J_1^2 - J_2 \tag{134}$$

$$\bar{J}'_3 = J_3 - \frac{1}{3} J_1 J_2 + \frac{2}{27} J_1^3 \tag{135}$$

Principal stress space
and octahedral stresses

39. Since the three principal stresses are orthogonal, they form a three-dimensional space called the principal stress space (Figure 5). Of particular interest in the principal stress space are the octahedral planes. The direction cosines of a normal to an octahedral plane are (Figure 5)

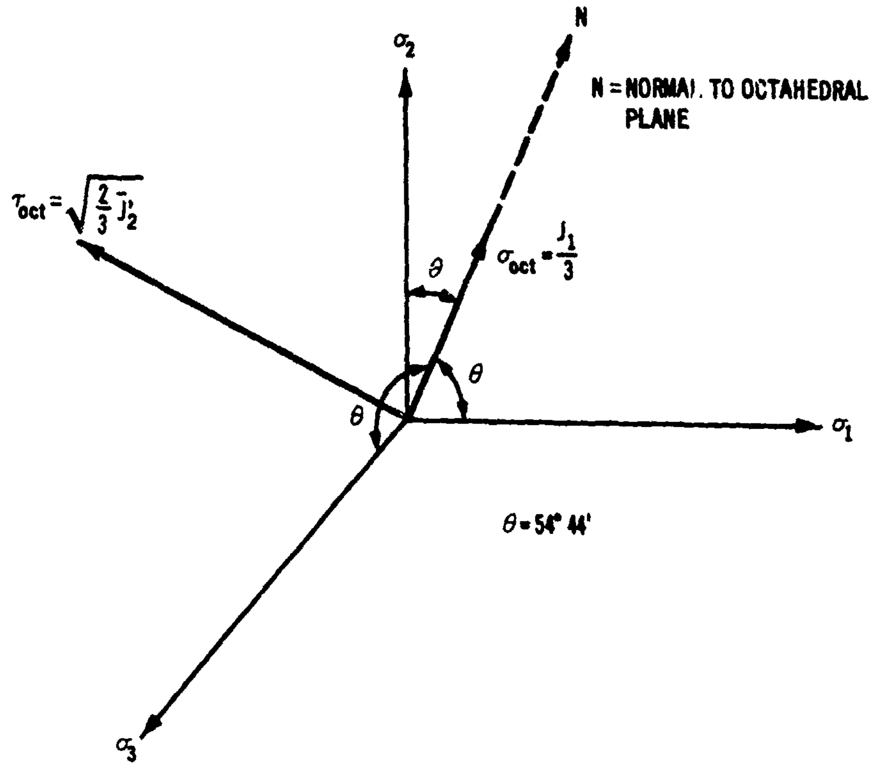



Figure 5. Principal stress space

$$\cos (N , \sigma_1) = \cos (N , \sigma_2) = \cos (N , \sigma_3) = \cos (54° 44')$$

$$= \frac{1}{\sqrt{3}} \qquad (136)$$

The normal and shear stresses on octahedral planes are denoted as $\sigma_{oct}$ and $\tau_{oct}$ , respectively. The magnitude of $\sigma_{oct}$ and $\tau_{oct}$ can be determined from the transformation law of stress tensor[2]

$$\sigma_{oct} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{Bmatrix} = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) \qquad (137)$$

and

$$\tau_{oct} = \sqrt{\left(\frac{\sigma_1}{\sqrt{3}}\right)^2 + \left(\frac{\sigma_2}{\sqrt{3}}\right)^2 + \left(\frac{\sigma_3}{\sqrt{3}}\right)^2 - \left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2}$$

$$= \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} \tag{138}$$

Using Equations 124 and 118 it can be shown that for a general state of stress

$$\sigma_{oct} = \frac{J_1}{3} \tag{139}$$

$$\tau_{oct} = \sqrt{\frac{2}{3} J_2'} \tag{140}$$

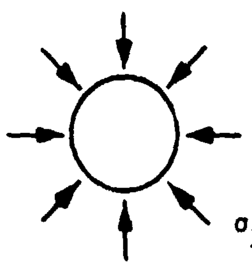
Equations 139 and 140 indicate that the octahedral stresses are also invariant. The octahedral space ($\tau_{oct}$ versus $\sigma_{oct}$) is commonly used for plotting stress paths for various laboratory tests. In this report we will use $\sqrt{3/2}\ \tau_{oct}$ versus $\sigma_{oct}$ space (i.e., $\sqrt{J_2'}$ versus $J_1/3$) for defining stress paths.

Examples of simple states of stress

40. The following states of stress are often utilized in the laboratory in order to determine the stress-strain properties of a material:

a. Spherical or hydrostatic state of stress.

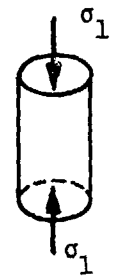
$$\sigma_{ij} = \frac{J_1}{3}\delta_{ij} = \begin{bmatrix} \frac{J_1}{3} & 0 & 0 \\ 0 & \frac{J_1}{3} & 0 \\ 0 & 0 & \frac{J_1}{3} \end{bmatrix}$$



$\sigma_1 = \sigma_2 = \sigma_3 = \frac{J_1}{3}$

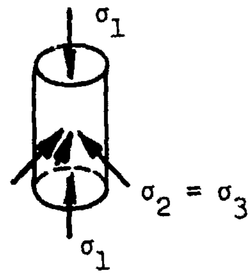
b. Uniaxial state of stress.

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



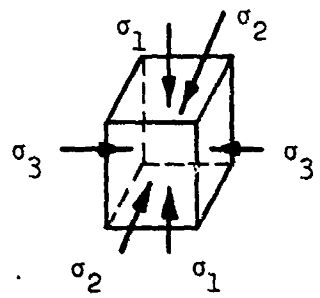

c. Cylindrical state of stress.

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_3 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



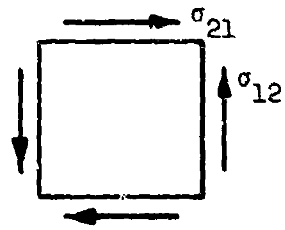

d. Triaxial state of stress.

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$




e. Pure shear.

$$\sigma_{ij} = \begin{bmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$




Note that in examples a through d all stresses are principal stresses. Stress paths associated with the above states of stress can be readily defined in the $\sqrt{\bar{J}_2'}$ versus $J_1/3$ space. The stress path associated with spherical or hydrostatic state of stress is shown in Figure 6a. It is noted that for spherical state of stress $\bar{J}_2'$ is zero. The stress path associated with uniaxial state of stress is shown in Figure 6b. For uniaxial state of stress $\sqrt{\bar{J}_2'} = \sigma_1/\sqrt{3}$ and $J_1/3 = \sigma_1/3$ resulting in the expression $\sqrt{\bar{J}_2'} = \sqrt{3}\ (J_1/3)$ for the stress path.

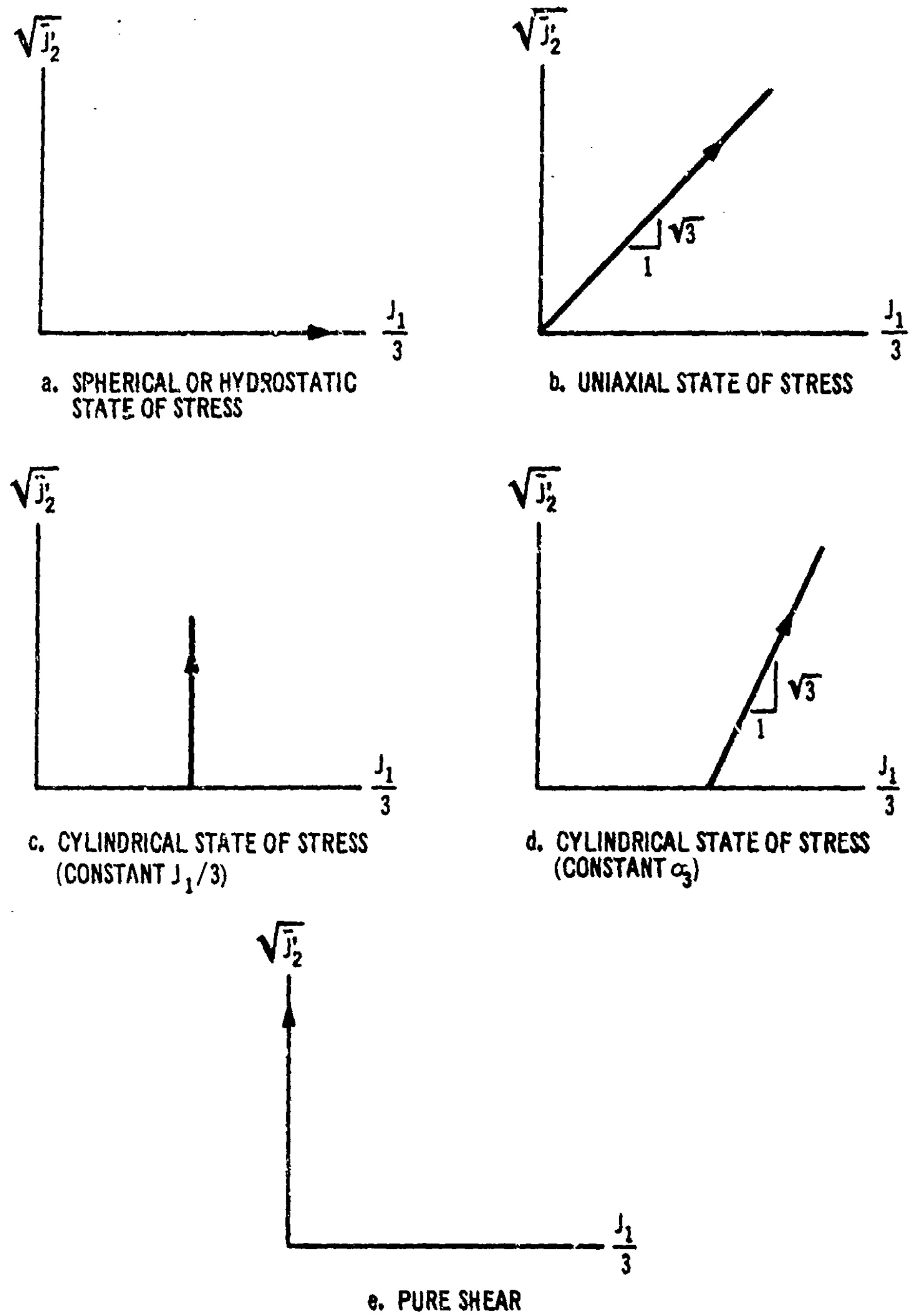



Figure 6. Stress paths associated with simple states of stress

Figures 6c and 6d depict special stress paths associated with cylindrical state of stress. In Figure 6c the material is first loaded hydrostatically and then sheared while $J_1/3$ is kept constant. In Figure 6d the material is first loaded hydrostatically and then sheared by increasing $\sigma_1$ while keeping $\sigma_3$ constant. Since for cylindrical state of stress $\sqrt{J_2'} = (\sigma_1 - \sigma_3)/\sqrt{3}$ and $J_1/3 = (\sigma_1 + 2\sigma_3)/3$ it follows that the expression for the stress path of Figure 6d becomes $\sqrt{J_2'} = \sqrt{3}\,(J_1/3 - \sigma_3)$ . The stress path associated with pure shear test is shown in Figure 6e. In the case of pure shear $J_1/3 = 0$ . In the actual laboratory coordinate system, the stress components $\sigma_1$ and $\sigma_2 = \sigma_3$ associated with cylindrical state of stress are usually denoted by $\sigma_a$ (axial stress) and $\sigma_r = \sigma_\theta$ (radial stress), respectively. In the case of triaxial state of stress, the stress components $\sigma_1$ , $\sigma_2$ , and $\sigma_3$ are denoted by $\sigma_x$ , $\sigma_y$ , and $\sigma_z$ , respectively. For pure shear the only nonzero stress component $\sigma_{12}$ is generally denoted by $\tau$ .

## Strain Tensor

41. Let us consider a cylindrical specimen of length $\ell_0$ and extend it to length $\ell$ . The ratio $\ell/\ell_0$ is defined as the stretch $\underset{\sim}{\lambda}$ .

$$\underset{\sim}{\lambda} = \ell/\ell_0 \tag{141}$$

The question is, what is the axial strain in the specimen? There are several measures of strain that can be used to determine the axial strain $\varepsilon$ in the specimen.[3] These measures, named after Cauchy, Green, Hencky, Almansi, and Swainger, respectively, are:

$$\varepsilon^C = \underset{\sim}{\lambda} - 1 \tag{142a}$$

$$\varepsilon^G = \frac{1}{2}\left(\underset{\sim}{\lambda}^2 - 1\right) \tag{142b}$$

$$\varepsilon^H = \ell n \underset{\sim}{\lambda} \tag{142c}$$

$$\varepsilon^A = \frac{1}{2}\left(1 - \frac{1}{\underset{\sim}{\lambda}^2}\right) \tag{142d}$$

$$\varepsilon^S = 1 - \frac{1}{\underset{\sim}{\lambda}} \tag{142e}$$

In order to demonstrate the difference between the various measures of strain given in Equation 142, let $\ell = 2\ell_0$ (i.e., let the length of specimen be doubled). The stretch $\underset{\sim}{\lambda} = 2$ in such case and from Equation 142 it follows that

$$\left.\begin{aligned} \varepsilon^C &= 100\% \\ \varepsilon^G &= 150\% \\ \varepsilon^H &= 69\% \\ \varepsilon^A &= 37.5\% \\ \varepsilon^S &= 50\% \end{aligned}\right\} \tag{143}$$

As observed from Equation 143, for a stretch of $\underset{\sim}{\lambda} = 2$, the difference between the various measures of strain is quite appreciable. Now let $\ell = 1.25\ell_0$, which gives a stretch of $\underset{\sim}{\lambda} = 1.25$. In view of Equation 142, for $\underset{\sim}{\lambda} = 1.25$ the various measures of strain become

$$\left.\begin{aligned} \varepsilon^C &= 25\% \\ \varepsilon^G &= 28\% \\ \varepsilon^H &= 22\% \\ \varepsilon^A &= 18\% \\ \varepsilon^S &= 20\% \end{aligned}\right\} \tag{144}$$

It is observed in this case that the difference between various measures of strain is not as appreciable as was the case for $\underset{\sim}{\lambda} = 2$. If the

stretch $\underset{\sim}{\lambda}$ is further reduced, say $\underset{\sim}{\lambda} = 1.1$ , Equation 142 will result in

$$\left.\begin{aligned} \varepsilon^C &= 10\% \\ \varepsilon^G &= 10.5\% \\ \varepsilon^H &= 9\% \\ \varepsilon^A &= 8.7\% \\ \varepsilon^S &= 9\% \end{aligned}\right\} \qquad (145)$$

Therefore, for small deformations (infinitesimal strain theory) the various measures of strain will yield approximately the same results. Our interest here is also within the framework of infinitesimal strain theory and we adopt the Cauchy measure of strain for further analysis.

42. In order to determine strain-displacement relations and define the infinitesimal strain tensor, we consider a particle $\overline{P}$ with position vector $\overline{x}_i$ in the $x_i$ coordinate system as shown in Figure 7.

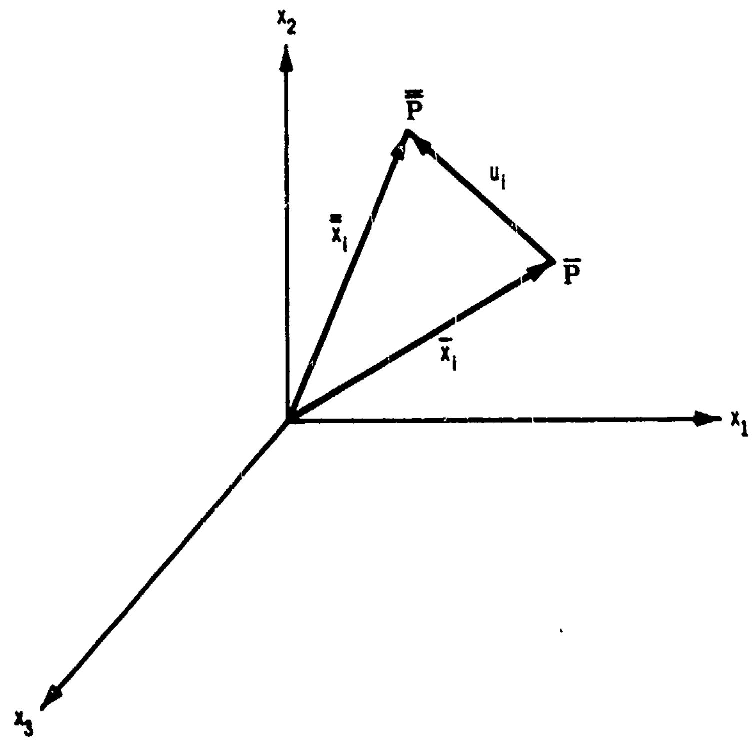



**Figure 7. Particle displacement in $x_i$ coordinate system**

We assume that the particle undergoes displacement $u_i$ and assumes a new position vector $\bar{\bar{x}}_i$ as depicted in Figure 7. From Figure 7 we can write

$$\bar{x}_i + u_i = \bar{\bar{x}}_i \tag{146}$$

or

$$u_i = \bar{\bar{x}}_i - \bar{x}_i \tag{147}$$

Since $\bar{\bar{x}}_i$ is a function of $\bar{x}_i$ , i.e., $\bar{\bar{x}}_i = \bar{\bar{x}}_i(\bar{x}_j)$ , we can differentiate Equation 146 with respect to $\bar{x}_j$ ; thus,

$$\delta_{ij} + u_{i,j} = \bar{\bar{x}}_{i,j} \tag{148}$$

The terms $\bar{\bar{x}}_{i,j}$ and $u_{i,j}$ are called the coordinate gradient and displacement gradient matrices, respectively. The displacement gradient matrix can be expressed as the sum of a symmetrical system and a skew-symmetrical system (see Equation 15)

$$u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{149}$$

The first term in Equation 149 is symmetrical and is called the infinitesimal strain tensor $\varepsilon_{ij}$ ; thus,

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{150}$$

The second term in Equation 149 is skew-symmetric and is called the rotation tensor $\Omega_{ij}$ ; thus,

$$\Omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{151}$$

Equation 150 relates the components of infinitesimal strain tensor with components of displacement vector.

43. To demonstrate the application of Equation 150, consider a rod of length $\ell_0$ extended to length $\ell$ as shown in Figure 8.

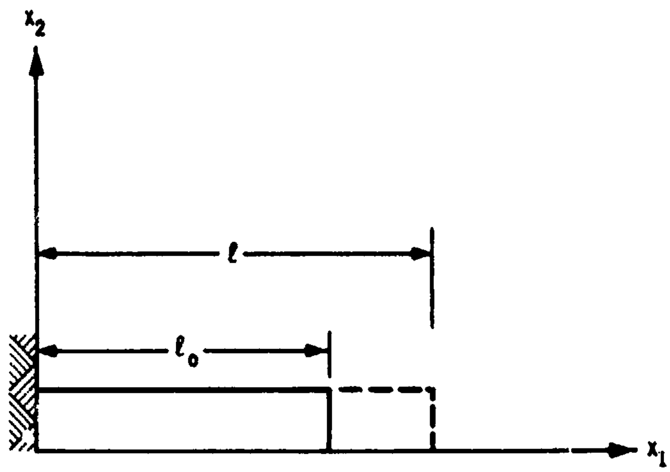



Figure 8. Rod in uniaxial extension

The boundary conditions associated with displacement $u_1$ in the $x_1$ direction are

$$\left.\begin{aligned} u_1 &= 0 && \text{at } x_1 = 0 \\ u_1 &= \ell - \ell_0 && \text{at } x_1 = \ell_0 \end{aligned}\right\} \tag{152}$$

For a homogeneous state of strain to exist in the rod, the displacement $u_1$ must be a linear function of $x_1$. Thus,

$$u_1 = Cx_1 \tag{153}$$

where $C$ is a constant. In view of Equation 152, Equation 153 becomes

$$u_1 = \frac{\ell - \ell_0}{\ell_0} x_1 \tag{154}$$

Substituting Equation 154 into Equation 150 we obtain

$$\epsilon_{11} = \frac{1}{2}(u_{1,1} + u_{1,1}) = u_{1,1} = \frac{\ell - \ell_0}{\ell_0} \tag{155}$$

which is the Cauchy measure of strain (see Equation 142a).

Invariants of strain tensor

44. Strain tensor is a second-order tensor and obeys the transformation law given in Equation 94, i.e.,

$$\varepsilon'_{ij} = a_{in}a_{jm}\varepsilon_{nm} \tag{156}$$

where $\varepsilon'_{ij}$ is referred to the $x'_i$ coordinate system. There are, therefore, three independent invariants associated with the strain tensor. As for the invariants of stress tensor, we define the invariants of strain tensor as

$$I_1 = I_\varepsilon = \varepsilon_{nn} \tag{157}$$

$$\bar{I}_2 = \frac{1}{2}\,\overline{II}_\varepsilon = \frac{1}{2}\,\varepsilon_{ik}\varepsilon_{ki} \tag{158}$$

$$\bar{I}_3 = \frac{1}{3}\,\overline{III}_\varepsilon = \frac{1}{3}\,\varepsilon_{ik}\varepsilon_{km}\varepsilon_{mi} \tag{159}$$

Strain deviation tensor

45. Strain tensor can be expressed as the sum of two symmetric tensors in the following manner

$$\varepsilon_{ij} = E_{ij} + \frac{1}{3}\,\varepsilon_{nn}\delta_{ij} \tag{160}$$

where the tensor

$$E_{ij} = \varepsilon_{ij} - \frac{1}{3}\,\varepsilon_{nn}\delta_{ij} \tag{161}$$

is referred to as the strain deviation tensor and $\varepsilon_{nn}\delta_{ij}/3$ is called the spherical strain tensor. As for invariants of stress deviation tensor, we define the invariants of strain deviation tensor as

$$\bar{I}'_2 = \frac{1}{2}\,\overline{II}_E = \frac{1}{2}\,E_{ik}E_{ki} \tag{162}$$

$$\overline{I}_3' = \frac{1}{3}\,\overline{III}_E = \frac{1}{3}\,E_{ik}E_{km}E_{mi} \tag{163}$$

The invariants of strain deviation tensor can also be expressed in terms of the invariants of strain tensor as follows:

$$\overline{I}_2' = \overline{I}_2 - \frac{1}{6}\,I_1^2 \tag{164}$$

$$\overline{I}_3' = \overline{I}_3 - \frac{2}{3}\,I_1\overline{I}_2 + \frac{2}{27}\,I_1^3 \tag{165}$$

Principal strains

46. The three principal values of strain tensor are referred to as principal strains and are denoted by (using the principal directions as reference axes)

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \tag{166}$$

The principal strains are the roots of the characteristic equation of strain tensor

$$\lambda^3 - I_\varepsilon\lambda^2 + II_\varepsilon\lambda - III_\varepsilon = 0 \tag{167}$$

where

$$I_\varepsilon = I_1 = \varepsilon_{nn} \tag{168a}$$

$$II_\varepsilon = \begin{vmatrix} \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{32} & \varepsilon_{33} \end{vmatrix} + \begin{vmatrix} \varepsilon_{11} & \varepsilon_{13} \\ \varepsilon_{31} & \varepsilon_{33} \end{vmatrix} + \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{vmatrix} \tag{168b}$$

$$III_\varepsilon = |\varepsilon| \tag{168c}$$

The two coefficients $II_\varepsilon$ and $III_\varepsilon$ are denoted by $I_2$ and $I_3$, respectively, and are related to the invariants $I_1$, $\bar{I}_2$, and $\bar{I}_3$ as follows:

$$I_2 = II_\varepsilon = \frac{1}{2}\left(I_1^2 - 2\bar{I}_2\right) \tag{169}$$

$$I_3 = III_\varepsilon = \bar{I}_3 - \bar{I}_2 I_1 + \frac{1}{6} I_1^3 \tag{170}$$

The invariants of strain deviation tensor can also be expressed in terms of $I_1$, $I_2$, and $I_3$:

$$\bar{I}_2' = \frac{1}{3} I_1^2 - I_2 \tag{171}$$

$$\bar{I}_3' = I_3 - \frac{1}{3} I_1 I_2 + \frac{2}{27} I_1^3 \tag{172}$$

Examples of simple
states of deformation

47. The following states of deformation are often utilized in the laboratory in order to determine the stress-strain properties of the material:

a. Uniform dilatation.

$$\varepsilon_{ij} = \frac{I_1}{3}\delta_{ij} = \begin{bmatrix} \frac{I_1}{3} & 0 & 0 \\ 0 & \frac{I_1}{3} & 0 \\ 0 & 0 & \frac{I_1}{3} \end{bmatrix}$$

b. Uniaxial state of strain.

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c. Cylindrical state of strain ($\varepsilon_2 = \varepsilon_3$).

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_2 \end{bmatrix}$$

d. Triaxial state of strain.

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

e. Simple shearing deformation (no volume change).

$$\varepsilon_{ij} = \begin{bmatrix} 0 & \varepsilon_{12} & 0 \\ \varepsilon_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that in examples a through d all strains are principal strains.

## Strain-Rate Tensor

48. The time derivative of infinitesimal strain tensor is referred to as rate of infinitesimal strain tensor, or simply strain-rate tensor, $\dot{\varepsilon}_{ij}$ ; thus,

$$\dot{\varepsilon}_{ij} = \frac{d}{dt} (\varepsilon_{ij}) \tag{173}$$

where $d/dt$ indicates differentiation with respect to time. In view of Equation 150, the strain-rate tensor takes the form

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \tag{174}$$

where $v_i$ = components of velocity vector.

### Invariants of strain-rate tensor

49. Strain-rate tensor is a second-order symmetric tensor and, like the stress and strain tensors, it obeys the transformation law given in Equation 94. Similarly, we define the invariants of strain-rate tensor as

$$\dot{I}_1 = I_{\dot{\varepsilon}} = \dot{\varepsilon}_{nn} \tag{175}$$

$$\dot{\bar{I}}_2 = \frac{1}{2}\overline{II}_{\dot{\varepsilon}} = \frac{1}{2}\dot{\varepsilon}_{ik}\dot{\varepsilon}_{ki} \tag{176}$$

$$\dot{\bar{I}}_3 = \frac{1}{3}\overline{III}_{\dot{\varepsilon}} = \frac{1}{3}\dot{\varepsilon}_{ik}\dot{\varepsilon}_{km}\dot{\varepsilon}_{mi} \tag{177}$$

### Strain-rate deviation tensor

50. Strain-rate tensor can be expressed as the sum of two symmetric tensors in the following manner

$$\dot{\varepsilon}_{ij} = \dot{E}_{ij} + \frac{1}{3}\dot{\varepsilon}_{nn}\delta_{ij} \tag{178}$$

where the tensor

$$\dot{E}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3}\dot{\varepsilon}_{nn}\delta_{ij} \tag{179}$$

is called the strain-rate deviation tensor and $\dot{\varepsilon}_{nn}\delta_{ij}/3$ is called the spherical strain-rate tensor. We define the invariants of strain-rate deviation tensor as

$$\dot{\bar{I}}_2' = \frac{1}{2}\,\overline{II}_{\dot{E}} = \frac{1}{2}\,\dot{E}_{ik}\dot{E}_{ki} \tag{180}$$

$$\dot{\bar{I}}_3' = \frac{1}{3}\,\overline{III}_{\dot{E}} = \frac{1}{3}\,\dot{E}_{ik}\dot{E}_{km}\dot{E}_{mi} \tag{181}$$

The invariants of strain-rate deviation tensor can also be expressed in terms of the invariants of strain-rate tensor:

$$\dot{\bar{I}}_2' = \dot{\bar{I}}_2 - \frac{1}{6}\,\dot{I}_1^2 \tag{182}$$

$$\dot{\bar{I}}_3' = \dot{\bar{I}}_3 - \frac{2}{3}\,\dot{I}_1\,\dot{\bar{I}}_2 + \frac{2}{27}\,\dot{I}_1^3 \tag{183}$$

Principal rates of strain

51. The three principal values of strain-rate tensor are denoted by

$$[\dot{\varepsilon}] = \begin{bmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{bmatrix} \tag{184}$$

and are called the principal rates of strain. The principal rates of strain are the roots of the characteristic equation of strain-rate tensor

$$\lambda^3 - I_{\dot{\varepsilon}}\lambda^2 + II_{\dot{\varepsilon}}\lambda - III_{\dot{\varepsilon}} = 0 \tag{185}$$

where

$$I_{\dot{\varepsilon}} = \dot{I}_1 = \dot{\varepsilon}_{nn} \tag{186a}$$

$$II_{\dot{\varepsilon}} = \begin{vmatrix} \dot{\varepsilon}_{22} & \dot{\varepsilon}_{23} \\ \dot{\varepsilon}_{32} & \dot{\varepsilon}_{33} \end{vmatrix} + \begin{vmatrix} \dot{\varepsilon}_{11} & \dot{\varepsilon}_{13} \\ \dot{\varepsilon}_{31} & \dot{\varepsilon}_{33} \end{vmatrix} + \begin{vmatrix} \dot{\varepsilon}_{11} & \dot{\varepsilon}_{12} \\ \dot{\varepsilon}_{21} & \dot{\varepsilon}_{22} \end{vmatrix} \qquad (186b)$$

$$III_{\dot{\varepsilon}} = |\dot{\varepsilon}| \qquad (186c)$$

The two coefficients $II_{\dot{\varepsilon}}$ and $III_{\dot{\varepsilon}}$ are denoted by $\dot{i}_2$ and $\dot{i}_3$, respectively, and are related to the invariants $\dot{i}_1$, $\dot{\bar{I}}_2$, and $\dot{\bar{I}}_3$ as follows:

$$\dot{i}_2 = II_{\dot{\varepsilon}} = \frac{1}{2}\left(\dot{i}_1^2 - 2\dot{\bar{I}}_2\right) \qquad (187)$$

$$\dot{i}_3 = III_{\dot{\varepsilon}} = \dot{\bar{I}}_3 - \dot{\bar{I}}_2\dot{i}_1 + \frac{1}{6}\dot{i}_1^3 \qquad (188)$$

The invariants of strain-rate deviation tensor can also be expressed in terms of $\dot{i}_1$, $\dot{i}_2$, and $\dot{i}_3$:

$$\dot{\bar{I}}_2' = \frac{1}{3}\dot{i}_1^2 - \dot{i}_2 \qquad (189)$$

$$\dot{\bar{I}}_3' = \dot{i}_3 - \frac{1}{3}\dot{i}_1\dot{i}_2 + \frac{2}{27}\dot{i}_1^3 \qquad (190)$$

## Equations of Continuity and Motion

52. The motion of any continuum is governed by the following laws:

a. Conservation of mass.

b. Conservation of energy.

c. Balance of linear momentum.

d. Balance of angular momentum.

e. Principle of inadmissibility of decreasing entropy.

These laws constitute the basic axioms of continuum mechanics.[4] In the absence of distributed couples, the balance of angular momentum leads to the symmetry of stress tensor, $\sigma_{ij} = \sigma_{ji}$ . If mechanical energy is the only form of energy to be considered in a problem (as is the case in this report), the above principles lead to the continuity equation

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0 \tag{191}$$

and the equations of motion

$$\sigma_{ij,j} + f_i = \rho a_i \tag{192}$$

where $\rho$ = mass density , $v_i$ = components of velocity vector , $f_i$ = components of body force , and $a_i$ = components of acceleration vector . Equations 191 and 192 are applicable to all materials.

## Constitutive Equations

53. Equations 191 and 192 constitute four equations that involve ten unknown functions of time and space: the mass density $\rho$ , the three velocity components $v_i$ , and the six independent stress components $\sigma_{ij}$ . The body force components $f_i$ are known quantities and the acceleration components $a_i$ are expressible in terms of the velocity components $v_i$ . Obviously, Equations 191 and 192 are inadequate to determine the motion or deformation of a medium subjected to external disturbances, such as surface forces. Therefore, six additional equations relating the ten unknown variables $\rho$ , $v_i$ , and $\sigma_{ij}$ are required. Such relationships are referred to as constitutive equations, which relate the stress tensor $\sigma_{ij}$ to deformation or motion of the medium. As was pointed out previously, Equations 191 and 192 are applicable to all materials, whereas constitutive equations represent the intrinsic response of a particular material. Furthermore, a constitutive equation provides a mathematical description or definition of an ideal material rather than a statement of a universal law. The

general form of a constitutive equation may be expressed by the functional form (considering only mechanical effects)

$$f_{ij}\ (v_k\ ,\ \sigma_{mn}\ ,\ \rho) = 0 \tag{193}$$

or

$$g_{ij}(\varepsilon_{rs}\ ,\ \dot{\varepsilon}_{mn}\ ,\ \sigma_{ab}\ ,\ \dot{\sigma}_{cd}\ ,\ \rho) = 0 \tag{194}$$

where $\dot{\sigma}_{cd}$ = time derivative of stress tensor . Equations 191, 192, and 194 (or Equation 193) , therefore, constitute ten equations in ten unknowns and will lead, in conjunction with kinematic relations given by Equations 150 and 174, to a complete description of the boundary-value problem. In addition to the above-mentioned equations, boundary conditions in terms of boundary displacement and/or surface forces must also be specified to completely define a particular problem of interest.

54. In order for constitutive equations to describe physical materials adequately, the functional forms $f_{ij}$ or $g_{ij}$ must remain invariant with respect to rigid motion of spatial coordinate $x_i$ . This requirement stems from the fact that the response of a material is independent of the motion of the observer. Furthermore, the functionals $f_{ij}$ or $g_{ij}$ must be consistent with the general principles of conservation or balance of mass, momentum, and energy.

55. We adopt Equation 194, relating four second-order symmetric tensors $\varepsilon_{rs}$ , $\dot{\varepsilon}_{mn}$ , $\sigma_{ab}$ , and $\dot{\sigma}_{cd}$ , as a basis for development of various constitutive equations in the following parts of this report.

## PART IV: CONSTITUTIVE EQUATIONS OF ELASTIC MATERIALS

56. For an elastic material, the state of stress is a function of the current state of strain only. Furthermore, an elastic material returns to its initial state after a load-unload cycle of deformation (no permanent strain). The stress tensor can, therefore, be expressed in terms of strain tensor

$$\sigma_{ij} = F_{ij}(\varepsilon_{mn}) \tag{195}$$

where $F_{ij}$ = elastic response function. Two different procedures have been utilized in order to determine the response function $F_{ij}$ for isotropic materials. The first procedure, referred to as Cauchy's method, is based on the Cayley-Hamilton theorem (Equation 59). The second procedure, referred to as Green's method, is based on conservation of energy. Both of these methods are dealt with in this part of the report.

### Cauchy's Method

57. The response function $F_{ij}$ in Equation 195 can be expanded as a polynomial in the strain tensor $\varepsilon_{ij}$, i.e.,

$$\sigma_{ij} = a_0 + a_1\varepsilon_{ij} + a_2\varepsilon_{im}\varepsilon_{mj} + a_3\varepsilon_{im}\varepsilon_{mn}\varepsilon_{nj} + \ldots \tag{196}$$

where $a_0$, $a_1$, ... $a_n$ are real coefficients. Utilizing the Cayley-Hamilton theorem we can express Equation 196 in the following form (see Equations 62 and 63)

$$\sigma_{ij} = \phi_0\delta_{ij} + \phi_1\varepsilon_{ij} + \phi_2\varepsilon_{im}\varepsilon_{mj} \tag{197}$$

where $\phi_0$, $\phi_1$, and $\phi_2$ are elastic response coefficients which are polynomial functions of strain invariants. Equation 197 is referred to as the Cauchy elastic constitutive equation. Alternately, for an elastic material we can express

$$\varepsilon_{ij} = \Psi_0\delta_{ij} + \Psi_1\sigma_{ij} + \Psi_2\sigma_{im}\sigma_{mj} \tag{198}$$

where $\Psi_0$ , $\Psi_1$ , and $\Psi_2$ are elastic response coefficients which are polynomial functions of stress invariants. From Equation 197 it follows that for isotropic elastic materials the initial state of stress is hydrostatic, i.e.,

$$\sigma_{ij} = \phi_0\delta_{ij} \quad \text{when} \quad \varepsilon_{ij} = 0 \tag{199}$$

Also, using the transformation law of a second-order tensor (Equation 94), it can be shown that Equation 197 is form invariant with respect to rigid motion of a spatial coordinate system, i.e.,

$$\begin{aligned} \sigma'_{mn} &= a_{mi}a_{nj}\sigma_{ij} \\ &= \phi_0 a_{mi}a_{nj}\delta_{ij} + \phi_1 a_{mi}a_{nj}\varepsilon_{ij} + \phi_2 a_{mi}a_{nj}\varepsilon_{ik}\varepsilon_{kj} \\ &= \phi_0\delta'_{mn} + \phi_1\varepsilon'_{mn} + \phi_2\varepsilon'_{mk}\varepsilon'_{kn} \end{aligned} \tag{200}$$

where $\varepsilon'_{mn}$ is referred to the primed (rotated) coordinate system. We can now utilize Equation 197 to develop various types of isotropic elastic constitutive equations.

Linear elastic material

58. For linear elastic materials the response coefficient $\phi_2$ vanishes. The response coefficient $\phi_1$ is a constant and $\phi_0$ is a linear function of the first strain invariant. Assuming that the initial state of stress is zero, the constitutive equation of linear elastic material can be written as

$$\sigma_{ij} = AI_1\delta_{ij} + B\varepsilon_{ij} \tag{201}$$

where A and B are material constants. In order to determine the physical meaning of the material constants A and B let us consider a simple shearing deformation defined by

$$\varepsilon_{ij} = \begin{bmatrix} 0 & \varepsilon_{12} & 0 \\ \varepsilon_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{202}$$

For this state of deformation, Equation 201 reduces to

$$\sigma_{ij} = \begin{bmatrix} 0 & B\varepsilon_{12} & 0 \\ B\varepsilon_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{203}$$

Since $\varepsilon_{12}$ is half the shearing strain (see Equation 150), it follows that $B$ is two times the shear modulus which we define as $G$ ; thus,

$$B = 2G \tag{204}$$

Next, we consider uniform dilatation defined by

$$\varepsilon_{ij} = \frac{I_1}{3}\,\delta_{ij} = \begin{bmatrix} \frac{I_1}{3} & 0 & 0 \\ 0 & \frac{I_1}{3} & 0 \\ 0 & 0 & \frac{I_1}{3} \end{bmatrix} \tag{205}$$

For this state of deformation, Equation 201 becomes (invoking Equation 204)

$$\sigma_{ij} = \left(AI_1 + 2G\,\frac{I_1}{3}\right)\delta_{ij} \tag{206}$$

Taking the trace of $\sigma_{ij}$ (let $i = j$) we obtain

$$\frac{\sigma_{ii}}{3} = \frac{J_1}{3} = \left(A + \frac{2G}{3}\right)I_1 \tag{207}$$

Equation 207 relates pressure ($J_1/3$) to volumetric strain ($I_1$). The slope of the pressure-volumetric strain relation is defined as bulk modulus $K$ ; thus,

$$A + \frac{2G}{3} = K \tag{208}$$

or

$$A = K - \frac{2G}{3} \tag{209}$$

The constant $A$ is usually denoted as $\lambda$ and is referred to as the Lame constant. In terms of the shear and bulk moduli, the constitutive equation of linear elastic material (Equation 201) then becomes

$$\sigma_{ij} = KI_1\delta_{ij} + 2G\left(\varepsilon_{ij} - \frac{I_1}{3}\delta_{ij}\right) \tag{210}$$

The expression $\varepsilon_{ij} - I_1\delta_{ij}/3$ is recognized as the strain deviation tensor $E_{ij}$ (Equation 161); thus,

$$\sigma_{ij} = KI_1\delta_{ij} + 2GE_{ij} \tag{211}$$

From Equations 122, 207, and 208, it follows that the constitutive equation of linear elastic materials can also be written as

$$S_{ij} = 2GE_{ij} \tag{212a}$$

$$\frac{J_1}{3} = KI_1 \tag{212b}$$

Equation 212 indicates that for linear elastic materials volumetric strain is caused by hydrostatic stress only, and that the shearing response of the material is independent of pressure.

59. Using Equation 212 we can readily express the strain tensor in terms of stress tensor

$$E_{ij} = \frac{S_{ij}}{2G} \tag{213a}$$

$$I_1 = \frac{J_1}{3K} \tag{213b}$$

or, using Equation 161,

$$\varepsilon_{ij} = \frac{J_1}{9K}\,\delta_{ij} + \frac{S_{ij}}{2G} \tag{214}$$

60. We will now proceed to examine the behavior of linear elastic materials under various states of stress and deformation. Let us first consider uniaxial state of stress, a common laboratory test, defined by

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{215}$$

For this state of stress, Equation 214 results in

$$\varepsilon_{ij} = \begin{bmatrix} \left(\frac{3K + G}{9KG}\right)\sigma_1 & 0 & 0 \\ 0 & \left(\frac{2G - 3K}{18KG}\right)\sigma_1 & 0 \\ 0 & 0 & \left(\frac{2G - 3K}{18KG}\right)\sigma_1 \end{bmatrix} \tag{216}$$

Equation 216 indicates that under uniaxial state of stress

$$\sigma_1 = \frac{9KG}{3K + G}\,\varepsilon_1 \tag{217}$$

$$\varepsilon_2 = \varepsilon_3 = -\left(\frac{3K - 2G}{6K + 2G}\right)\varepsilon_1 \tag{218}$$

The ratio $\sigma_1/\varepsilon_1$ under uniaxial state of stress is referred to as Young's modulus $E$, and the ratio of radial strain to axial strain is called Poisson's ratio $\nu$; thus (for an incompressible elastic material $\nu = 1/2$)

$$E = \frac{9KG}{3K + G} \tag{219}$$

$$\nu = \frac{3K - 2G}{6K + 2G} \tag{220}$$

61. Another common laboratory test is the uniaxial strain test defined by

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{221}$$

For this state of deformation, Equation 211 results in

$$\sigma_{ij} = \begin{bmatrix} (K + 4G/3)\varepsilon_1 & 0 & 0 \\ 0 & (K - 2G/3)\varepsilon_1 & 0 \\ 0 & 0 & (K - 2G/3)\varepsilon_1 \end{bmatrix} \tag{222}$$

From Equations 222, 219, and 220 it follows that under uniaxial strain condition

$$\sigma_1 = (K + 4G/3)\varepsilon_1 = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)}\,\varepsilon_1 \tag{223}$$

$$\sigma_2 = \sigma_3 = \left(\frac{3K - 2G}{3K + 4G}\right)\sigma_1 = \left(\frac{\nu}{1 - \nu}\right)\sigma_1 \tag{224}$$

It is noted that $\sigma_2$ is the radial stress required to prevent radial strain. The ratio $\sigma_1/\varepsilon_1$ under uniaxial state of strain is referred to as the constrained modulus $M$ ; thus,

$$M = K + 4G/3 = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \tag{225}$$

Using Equations 223 and 224 we can determine an expression for the stress path associated with the uniaxial state of strain in the $\sqrt{J_2'}$ versus $J_1/3$ space:

$$\sqrt{J_2'} = \frac{2G}{\sqrt{3}K} J_1/3 \tag{226}$$

In terms of Poisson's ratio $\nu$ , the equation of the stress path becomes

$$\sqrt{J_2'} = \frac{\sqrt{3}(1 - 2\nu)}{(1 + \nu)} J_1/3 \tag{227}$$

62. Next, let us consider the behavior of linear elastic materials under condition of plane strain defined by

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{228}$$

For condition of plane strain, Equation 211 results in

$$\sigma_{ij} = \begin{bmatrix} \left(K + \frac{4}{3}G\right)\varepsilon_{11} + \left(K - \frac{2G}{3}\right)\varepsilon_{22} & 2G\varepsilon_{12} & 0 \\ 2G\varepsilon_{21} & \left(K - \frac{2G}{3}\right)\varepsilon_{11} + \left(K + \frac{4}{3}G\right)\varepsilon_{22} & 0 \\ 0 & 0 & \left(K - \frac{2G}{3}\right)\left(\varepsilon_{11} + \varepsilon_{22}\right) \end{bmatrix} \tag{229}$$

From Equation 229 it follows that

$$\sigma_{33} = \left(\frac{3K - 2G}{6K + 2G}\right)(\sigma_{11} + \sigma_{22}) = \nu(\sigma_{11} + \sigma_{22}) \tag{230}$$

Equation 230 gives the magnitude of stress $\sigma_{33}$ necessary to maintain plane strain condition, i.e., $\varepsilon_{33} = 0$ .

63. The counterpart of plane strain is the condition of plane stress defined by

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{231}$$

For plane stress condition, Equation 214 results in

$$\varepsilon_{ij} = \begin{bmatrix} \left(\frac{1}{9K} + \frac{1}{3G}\right)\sigma_{11} + \left(\frac{1}{9K} - \frac{1}{6G}\right)\sigma_{22} & \frac{\sigma_{12}}{2G} & 0 \\ \frac{\sigma_{21}}{2G} & \left(\frac{1}{9K} - \frac{1}{6G}\right)\sigma_{11} + \left(\frac{1}{9K} + \frac{1}{3G}\right)\sigma_{22} & 0 \\ 0 & 0 & \left(\frac{1}{9K} - \frac{1}{6G}\right)(\sigma_{11} + \sigma_{22}) \end{bmatrix} \tag{232}$$

From Equation 232 it follows that

$$\varepsilon_{33} = \left(\frac{1}{9K} - \frac{1}{6G}\right)(\sigma_{11} + \sigma_{22}) = -\frac{\nu}{1 - \nu}(\varepsilon_{11} + \varepsilon_{22}) \tag{233}$$

Equation 233 gives the magnitude of strain $\varepsilon_{33}$ produced by condition of plane stress, i.e., $\sigma_{33} = 0$ .

64. The constitutive equations of linear elastic material expressed in terms of various combinations of elastic constants are given in Table 1 for ready use.

Table 1

<u>Constitutive Equations of Linear Elastic material Expressed in Terms of Various Combinations of Elastic Constants</u>

| Elastic Constants | Young's Modulus, $E$ | Poisson's Ratio, $\nu$ | Shear Modulus, $G$ |
|---|---|---|---|
| $E$ | | $\sigma_{ij} = \frac{\nu E}{(1+\nu)(1-2\nu)} I_1\delta_{ij} + \frac{E}{1+\nu}\epsilon_{ij}$ | $\sigma_{ij} = 2G\epsilon_{ij} + \frac{GE - 2G^2}{3G - E} I_1\delta_{ij}$ |
| | | $\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E} J_1\delta_{ij}$ | $\epsilon_{ij} = \frac{\sigma_{ij}}{2G} - \left(\frac{1}{2G} - \frac{1}{E}\right) J_1\delta_{ij}$ |
| $\nu$ | | | $\sigma_{ij} = 2G\epsilon_{ij} + \frac{2G\nu}{1-2\nu} I_1\delta_{ij}$ |
| | | | $\epsilon_{ij} = \frac{\sigma_{ij}}{2G} - \frac{\nu}{2G(1+\nu)} J_1\delta_{ij}$ |
| Bulk Modulus, $K$ | $\sigma_{ij} = \frac{6KE}{9K - E}\epsilon_{ij} + \frac{9K^2 - 3KE}{9K - E} I_1\delta_{ij}$ | $\sigma_{ij} = \frac{3\nu K}{1+\nu} I_1\delta_{ij} + \frac{3K(1-2\nu)}{1+\nu}\epsilon_{ij}$ | $\sigma_{ij} = 2G\epsilon_{ij} + \left(K - \frac{2G}{3}\right) I_1\delta_{ij}$ |
| | $\epsilon_{ij} = \frac{9K - E}{6KE}\sigma_{ij} - \frac{3K - E}{6KE} J_1\delta_{ij}$ | $\epsilon_{ij} = \frac{1+\nu}{3K(1-2\nu)}\sigma_{ij} - \frac{\nu}{3K(1-2\nu)} J_1\delta_{ij}$ | $\epsilon_{ij} = \frac{\sigma_{ij}}{2G} + \left(\frac{1}{9K} - \frac{1}{6G}\right) J_1\delta_{ij}$ |

## Nonlinear elastic material

65. Constitutive equations for various classes of nonlinear elastic material can be developed from the general form of the Cauchy elastic constitutive equation (Equations 197 and 198). Before we develop constitutive equations for various classes of nonlinear elastic material it would be beneficial to examine the significance of the second-order terms $\varepsilon_{im}\varepsilon_{mj}$ and $\sigma_{im}\sigma_{mj}$ in Equations 197 and 198. Consider a simple shearing deformation of amount $2\gamma$ defined by the following strain tensor

$$\varepsilon_{ij} = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{234}$$

For this state of deformation, Equation 197 results in the following expression for the stress tensor

$$\sigma_{ij} = \phi_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \phi_1 \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \phi_2 \begin{bmatrix} \gamma^2 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{235}$$

From Equation 235, the shearing stress and shearing strain are related by

$$\sigma_{12} = \phi_1 \gamma \tag{236}$$

and the normal stresses are given as

$$\sigma_{11} = \sigma_{22} = \phi_0 + \phi_2 \gamma^2 \tag{237a}$$

$$\sigma_{33} = \phi_0 \tag{237b}$$

Equation 237 indicates that to maintain a simple shearing deformation

(Equation 234), normal stresses must be applied to the boundaries of the specimen. Since two of the normal stresses are unequal, Equation 237 predicts the occurrence of normal deviatoric stresses

$$S_{11} = S_{22} = \frac{1}{3}\phi_2\gamma^2 \tag{238}$$

on the shearing planes. This is a direct consequence of the second-order term $\varepsilon_{im}\varepsilon_{mj}$ in Equation 197 and is a departure from the linear theory where $\phi_2 = 0$. We now consider the counterpart of simple shearing deformation, i.e., simple shearing stress, and show that Equation 198 will predict volume change for this state of stress. Consider a simple shearing stress of amount $\tau$ defined by the following stress tensor

$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{239}$$

For this state of stress, Equation 198 results in the following expression for the strain tensor

$$\varepsilon_{ij} = \Psi_0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \Psi_1\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Psi_2\begin{bmatrix} \tau^2 & 0 & 0 \\ 0 & \tau^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{240}$$

From Equation 240 it follows that

$$\varepsilon_{ii} = I_1 = 3\Psi_0 + 2\Psi_2\tau^2 \tag{241}$$

which indicates that simple shearing stress is accompanied by volume change. Also, from Equation 240 it follows that there are normal deviatoric strains

$$E_{11} = E_{22} = \frac{1}{3} \Psi_2 \tau^2 \tag{242}$$

associated with the volume change. The occurrence of deviatoric strains is a direct consequence of the second-order term $\sigma_{im}\sigma_{mj}$ in Equation 198. From these two examples we can conclude that in the case of nonlinear elastic materials volumetric strains are caused by both the hydrostatic and shearing stresses. Also, the shearing response of the material is dependent on the hydrostatic state of stress. In order to further demonstrate these coupling effects we consider a combined state of hydrostatic and simple shearing stress given by the following stress tensor

$$\sigma_{ij} = \begin{bmatrix} P & \tau & 0 \\ \tau & P & 0 \\ 0 & 0 & P \end{bmatrix} \tag{243}$$

where $P$ = superimposed hydrostatic stress. For this state of stress, Equation 198 results in the following expression for the strain tensor

$$\varepsilon_{ij} = \Psi_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \Psi_1 \begin{bmatrix} P & \tau & 0 \\ \tau & P & 0 \\ 0 & 0 & P \end{bmatrix} + \Psi_2 \begin{bmatrix} P^2 + \tau^2 & 2P\tau & 0 \\ 2P\tau & P^2 + \tau^2 & 0 \\ 0 & 0 & P^2 \end{bmatrix} \tag{244}$$

The volumetric strain $I_1$ then becomes

$$\varepsilon_{ii} = I_1 = 3\Psi_0 + 3\Psi_1 P + \Psi_2(3P^2 + 2\tau^2) \tag{245}$$

and the shearing strain $\varepsilon_{12}$ takes the form

$$\varepsilon_{12} = \Psi_1 \tau + 2\Psi_2 P\tau \tag{246}$$

Equations 245 and 246 once again illustrate the coupling which exists

between the hydrostatic stress (or volumetric response) and the shearing response of nonlinear elastic materials.

66. We will now proceed to develop constitutive equations for various classes of nonlinear elastic material within the framework of the Cauchy elastic constitutive equation (Equations 197 and 198). The simplest form of nonlinear elastic material is the second-order stress-strain relation where terms in strain up to the second power are retained in the stress-strain relationship. To derive the constitutive relationship for second-order elastic material we will start from Equation 197 and express the response coefficients $\phi_0$, $\phi_1$, and $\phi_2$ in appropriate polynomials of strain invariants. Accordingly, for second-order material we can write

$$\phi_0 = C_1 I_1 + C_2 I_1^2 + C_3 \bar{I}_2 \tag{247a}$$

$$\phi_1 = C_4 + C_5 I_1 \tag{247b}$$

$$\phi_2 = C_6 \tag{247c}$$

where $C_1$ through $C_6$ are material constants which must be determined experimentally. Since the constitutive equations of second-order elastic material must degenerate to the first-order equation (Equation 210) if second power terms in strain are neglected, the material constants $C_1$ and $C_4$ should be replaced by $(K - 2G/3)$ and $2G$, respectively. The constitutive equation of second-order elastic material then becomes

$$\sigma_{ij} = \left[(K - 2G/3)I_1 + C_2 I_1^2 + C_3 \bar{I}_2\right]\delta_{ij} + (2G + C_5 I_1)\varepsilon_{ij} + C_6 \varepsilon_{im}\varepsilon_{mj} \tag{248}$$

67. Equation 248 contains six material constants. The physical meaning of these constants and the manner in which they can be determined from laboratory test results can be demonstrated by examining the behavior of second-order elastic materials under various states of stress and deformation. Let us first consider a simple shearing

deformation defined by Equation 202. For this state of deformation $I_1 = 0$ and $\overline{I}_2 = \varepsilon_{12}^2$, and Equation 248 results in the following relations for the components of stress tensor

$$\sigma_{12} = 2G\varepsilon_{12} \tag{249a}$$

$$\sigma_{11} = \sigma_{22} = (C_3 + C_6)\varepsilon_{12}^2 \tag{249b}$$

$$\sigma_{33} = C_3\varepsilon_{12}^2 \tag{249c}$$

Equation 249a indicates that a second-order elastic stress-strain relationship predicts a linear relation between shearing stress and shearing strain. Equations 249b and 249c give the magnitude of normal stresses, as a function of shearing strain, required to maintain shearing deformation. It is noted that the normal stresses are not uniform, thus resulting in normal deviatoric stresses

$$S_{\;\;} = S_{22} = \frac{1}{3} C_6\varepsilon_{12}^2 \tag{250}$$

on the shearing planes. The significance of the material constants $C_3$ and the combination $(C_3 + C_6)$ is realized from Equations 249c and 249b, respectively.

68. We next consider uniform dilatation defined by Equation 205. For this state of deformation $\overline{I}_2 = I_1^2/6$ and Equation 248 results in the following relationship between pressure and volumetric strain

$$\frac{J_1}{3} = KI_1 + \left(C_2 + \frac{1}{6} C_3 + \frac{1}{3} C_5 + \frac{1}{9} C_6\right)I_1^2 \tag{251}$$

Equation 251 describes a parabolic stress-strain relationship. The material constants $K$ and the combination $\left(C_2 + C_3/6 + C_5/3 + C_6/9\right)$ can be determined from experimental data by curve fitting techniques. It is noted that $K$ is the initial slope of the pressure-volumetric strain curve and is a positive constant. If the combination of the material constants in the parentheses is also positive, the stress-strain curve will be concave to the stress axis. If, on the other hand, this

combination is negative, the stress-strain curve will be concave to the strain axis.

69. We will next consider uniaxial state of strain defined by Equation 221. For this state of deformation $I_1 = \varepsilon_1$ and $\overline{I}_2 = \varepsilon_1^2/2$ and equation 248 results in

$$\sigma_1 = \left(K + \frac{4}{3} G\right)\varepsilon_1 + \left(C_2 + \frac{1}{2} C_3 + C_5 + C_6\right)\varepsilon_1^2 \tag{252}$$

$$\sigma_2 = \sigma_3 = \left(K - \frac{2}{3} G\right)\varepsilon_1 + \left(C_2 + \frac{C_3}{2}\right)\varepsilon_1^2 \tag{253}$$

Equations 252 and 253 also describe parabolic stress-strain relationships. The combinations of material constants $\left(C_2 + C_3/2 + C_5 + C_6\right)$ and $(C_2 + C_3/2)$ can be determined from experimental data by curve fitting techniques. As was pointed out previously, the shape of the stress-strain curves predic by Equations 252 and 253 depends on the sign of the combination of the material constants in the parentheses. The stress-strain curves will be concave to the stress axis if these combinations are positive, and concave to the strain axis if they are negative. Since $\varepsilon_1$ is the only nonvanishing strain component in uniaxial strain configuration, we can use Equations 252 and 253 to relate stress difference $\sigma_1 - \sigma_2$ to strain difference $\varepsilon_1$ , i.e.,

$$\sigma_1 - \sigma_2 = 2G\varepsilon_1 + (C_5 + C_6)\varepsilon_1^2 \tag{254}$$

Again, the shape of the stress-strain curve is determined from the sign of the combination $(C_5 + C_6)$ of the material constants.

70. More complicated states of stress and deformation, such as plane strain and triaxial stress conditions, can also be studied within the framework of second-order stress-strain law. Such states of stress generally lead to lengthy mathematical expressions between stress and strain components. For example, consider cylindrical state of strain defined by

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_2 \end{bmatrix} \tag{255}$$

For this state of deformation $I_1 = \varepsilon_1 + 2\varepsilon_2$ and $\bar{I}_2 = 1/2\left(\varepsilon_1^2 + 2\varepsilon_2^2\right)$ and Equation 248 results in the following expressions for the components of stress tensor

$$\sigma_1 = \left(K + \frac{4}{3} G\right)\varepsilon_1 + \left(C_2 + \frac{C_3}{2} + C_5 + C_6\right)\varepsilon_1^2 + \left(2K - \frac{4}{3} G\right)\varepsilon_2 + \left(4C_2 + C_3\right)\varepsilon_2^2 + \left(4C_2 + 2C_5\right)\varepsilon_1\varepsilon_2 \tag{256}$$

$$\sigma_2 = \sigma_3 = \left(K - \frac{2}{3} G\right)\varepsilon_1 + \left(C_2 + \frac{C_3}{2}\right)\varepsilon_1^2 + \left(2K + \frac{2G}{3}\right)\varepsilon_2 + \left(4C_2 + C_3 + 2C_5 + C_6\right)\varepsilon_2^2 + \left(4C_2 + C_5\right)\varepsilon_1\varepsilon_2 \tag{257}$$

Various stress paths may be employed in a laboratory test maintaining a cylindrical state of strain. The most common stress path used with this state of strain is to keep the lateral stress $\sigma_3$ constant while increasing $\sigma_1$ (Figure 6d). For this stress path it is possible, in principle, to solve for $\varepsilon_2$ in term of $\varepsilon_1$, using Equation 257 (since $\sigma_3$ is a constant), and then substitute the resulting expression into Equation 256 to develop a relationship between $\sigma_1$ and $\varepsilon_1$. Other stress paths such as constant $J_1/3$ path (Figure 6c) and uniaxial stress test (Figure 6b) can also be considered.

71. Following the same procedure we can develop and analyze more complicated nonlinear elastic stress-strain laws. Let us consider a third-order law where terms in strain up to the third power are retained in the stress-strain relationships. Accordingly, the response coefficients $\phi_0$, $\phi_1$, and $\phi_2$ (Equation 197) for a third-order stress-strain law take the following forms

$$\phi_0 = \left(K - \frac{2}{3} G\right) I_1 + C_2 I_1^2 + C_3 \overline{I}_2 + C_7 I_1^3 + C_8 I_1 \overline{I}_2 + C_9 \overline{I}_3 \qquad (258a)$$

$$\phi_1 = 2G + C_5 I_1 + C_{10} I_1^2 + C_{11} \overline{I}_2 \qquad (258b)$$

$$\phi_2 = C_6 + C_{12} I_1 \qquad (258c)$$

where $C_7$ through $C_{12}$ are six additional material constants which must be determined experimentally. A third-order elastic stress-strain law formulated within the framework of Equation 197 (Cauchy's method), therefore, contains twelve material constants. In view of Equation 197 the constitutive equation of third-order elastic material becomes

$$\sigma_{ij} = \left[\left(K - \frac{2}{3} G\right) I_1 + C_2 I_1^2 + C_3 \overline{I}_2 + C_7 I_1^3 + C_8 I_1 \overline{I}_2 + C_9 \overline{I}_3\right] \delta_{ij}$$
$$+ \left(2G + C_5 I_1 + C_{10} I_1^2 + C_{11} \overline{I}_2\right) \varepsilon_{ij} + (C_6 + C_{12} I_1) \varepsilon_{im} \varepsilon_{mj} \qquad (259)$$

It is noted that if third-order terms in strain are neglected, Equation 259 reduces to Equation 248 (constitutive equation of second-order elastic materials). It was pointed out previously that a second-order elastic stress-strain relationship predicts a linear relation between shearing stress and shearing strain. Nonlinear relation between shearing stress and shearing strain is due to third- or higher-order terms in strain tensor. This phenomenon, which is a departure from second-order effect (see Equation 249a), can be demonstrated by examining the behavior of third-order elastic materials under simple shearing deformation defined by Equation 202. For this state of deformation $I_1 = 0$, $\overline{I}_2 = \varepsilon_{12}^2$, and $\overline{I}_3 = 0$ and Equation 259 results in the following relation for the shearing stress $\sigma_{12}$

$$\sigma_{12} = 2G\varepsilon_{12} + C_{11} \varepsilon_{12}^3 \qquad (260)$$

Equation 260 is a third-order equation in shearing strain $\varepsilon_{12}$. The behavior of third-order stress-strain law under various states of stress

and deformation can also be studied similar to the second-order law. In the next section we will consider less complicated and perhaps more useful forms of elastic stress-strain laws referred to as quasi-linear elastic material.

Quasi-linear elastic material

72. Nonlinear elastic stress-strain laws are too complicated for application in all engineering problems. In many engineering problems only an approximate or gross behavior of the material under consideration needs to be modeled. For this reason, we will develop a number of simple stress-strain laws for simulating the gross behavior of a number of materials. We will start with Equation 197 (or Equation 198) by making the assumption that the response coefficient $\phi_2$ is zero. The basic constitutive equation then becomes

$$\sigma_{ij} = \phi_0 \delta_{ij} + \phi_1 \varepsilon_{ij} \tag{261}$$

where, as before, $\phi_0$ and $\phi_1$ are polynomial functions of strain invariants. Equation 261 is usually called a quasi-linear relation. From Equation 261 it follows that

$$\frac{J_1}{3} = \phi_0 + \frac{1}{3}\phi_1 I_1 \tag{262a}$$

$$S_{ij} = \phi_1 E_{ij} \tag{262b}$$

In view of Equations 124, 162, and 262 we can write

$$\phi_1 = \sqrt{\frac{\bar{J}_2'}{\bar{I}_2'}} \tag{263a}$$

$$\phi_0 = \frac{J_1}{3} - \frac{1}{3} I_1 \sqrt{\frac{\bar{J}_2'}{\bar{I}_2'}} \tag{263b}$$

Substituting Equation 263 in Equation 261 results in the following general constitutive equation for quasi-linear elastic material

$$\sigma_{ij} = \frac{J_1}{3} \delta_{ij} + \sqrt{\frac{\overline{J}_2'}{\overline{I}_2'}} \left(\varepsilon_{ij} - \frac{I_1}{3} \delta_{ij}\right) \tag{264}$$

It is only necessary to postulate (based on experimental evidence) mathematical expressions for $J_1$ and $\overline{J}_2'$ in terms of strain invariants in order to utilize Equation 264 for any material of interest. The inverse of Equation 264, resulting in strain-stress law, can be obtained from Equation 198 by assuming that $\Psi_2$ is zero and following the above procedure. The resulting relationship becomes

$$\varepsilon_{ij} = \frac{I_1}{3} \delta_{ij} + \sqrt{\frac{\overline{I}_2'}{\overline{J}_2'}} \left(\sigma_{ij} - \frac{J_1}{3} \delta_{ij}\right) \tag{265}$$

To use Equation 265 we need to express $I_1$ and $\overline{I}_2'$ in terms of stress invariants. For example, in the case of the linear elastic material:

$$\sqrt{\overline{J}_2'} = 2G\sqrt{\overline{I}_2'} \tag{266a}$$

$$\frac{J_1}{3} = KI_1 \tag{266b}$$

and Equations 264 and 265 reduce to constitutive equations of linear elastic materials (Equations 210 and 214, respectively).

73. We will now proceed to develop constitutive equations for various classes of quasi-linear elastic materials which are of interest for engineering application. For the simplest class of quasi-linear elastic material, in which there are no couplings between the deviatoric and volumetric responses of the material, we can write

$$\frac{J_1}{3} = f_1(I_1) \tag{267a}$$

$$\sqrt{\overline{J}_2'} = f_2\left(\sqrt{\overline{I}_2'}\right) \tag{267b}$$

where the functions $f_1$ and $f_2$ must be determined based on experimental evidence. A good approximation for a number of materials (such as clay soils) is to assume the following relations for $f_1$ and $f_2$

$$\frac{J_1}{3} = (P_0 + P_B)e^{\alpha I_1} - P_B \tag{268a}$$

$$\sqrt{J_2'} = \frac{\sqrt{I_2'}}{k_1 + k_2\sqrt{I_2'}} \tag{268b}$$

where $P_B$ , $\alpha$ , $k_1$ , and $k_2$ are material constants which must be determined experimentally. Equations 268a and 268b are depicted graphically in Figure 9. It is observed from Figure 9 that $P_0$ defines an initial hydrostatic state of stress (for materials that can sustain tension, $P_0$ can be taken to be zero) and $P_B$ defines the maximum hydrostatic tensile stress that the material can sustain before it fails (breaks) at such a tension. For materials that cannot sustain tension, the material constant $P_B$ is zero. In this case $P_0$ defines the state of "ease" or the initial stress state of the material. The material constant $k_1$ is proportional to the inverse of the initial shear modulus and $k_2$ is the inverse of the ultimate shear strength of the material. Substituting Equations 268a and 268b in Equation 264 results in the following quasi-linear elastic constitutive equation

$$\sigma_{ij} = \left[(P_0 + P_B)e^{\alpha I_1} - P_B\right]\delta_{ij} + \frac{1}{k_1 + k_2\sqrt{I_2'}}\left(\epsilon_{ij} - \frac{I_1}{3}\delta_{ij}\right) \tag{269}$$

The inverse of Equation 269 resulting in strain-stress law can be obtained by inverting Equations 268a and 268b and substituting the resulting expressions for $I_1$ and $\sqrt{I_2'}$ into Equation 265 as follows:

$$\epsilon_{ij} = \frac{1}{3}\left[\frac{1}{\alpha}\ln\left(\frac{J_1/3 + P_B}{P_0 + P_B}\right)\right]\delta_{ij} + \frac{k_1}{1 - k_2\sqrt{J_2'}}\left(\sigma_{ij} - \frac{J_1}{3}\delta_{ij}\right) \tag{270}$$

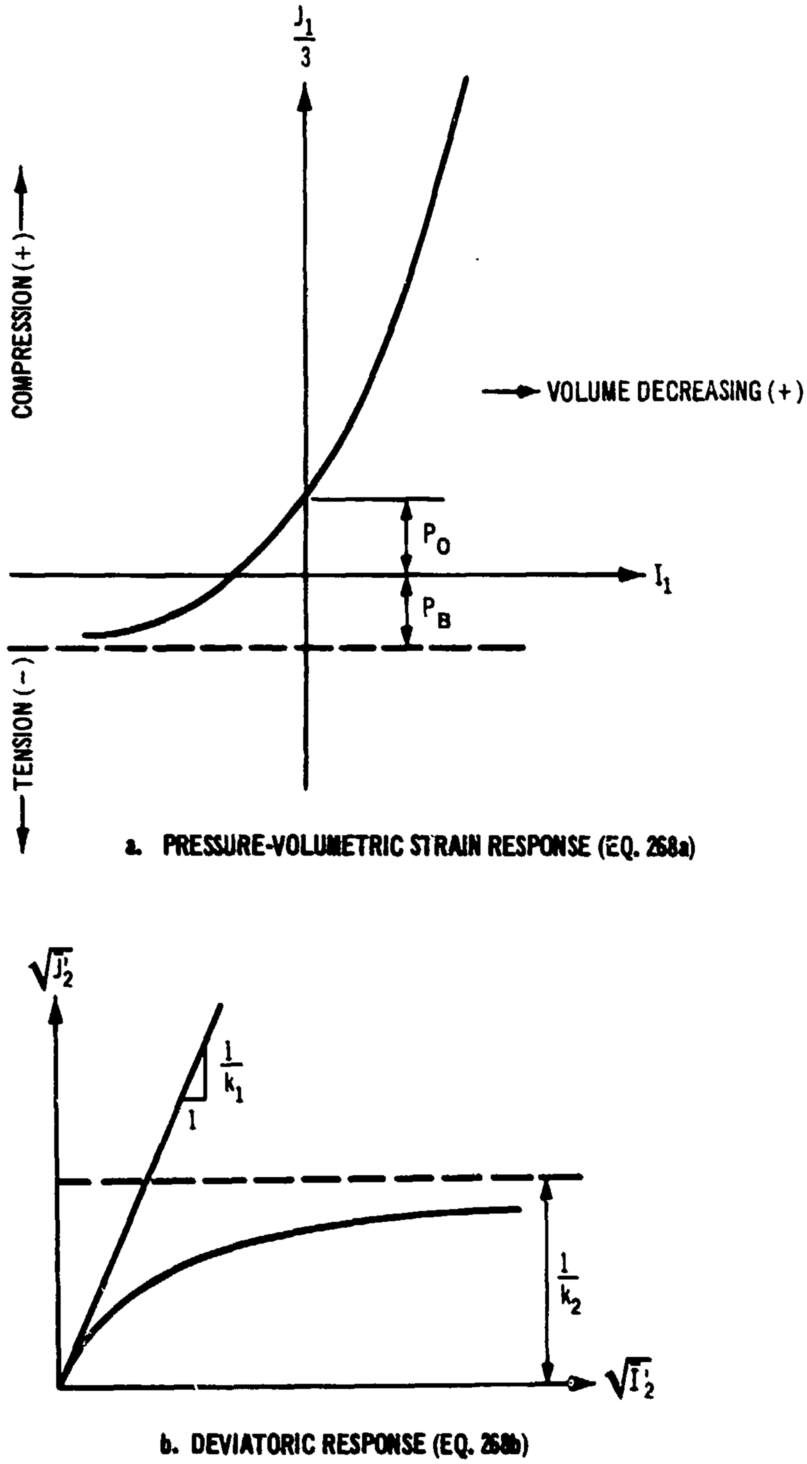



Figure 9. Assumed relationships for quasi-linear elastic material

74. Equation 269 (or Equation 270) is a simple but useful constitutive equation which can be used to study the stress-strain behavior of a number of physically nonlinear materials. It contains only five material constants which have physical meaning and can easily be determined experimentally. Having determined the numerical values of these material constants we can use Equations 269 and 270 to predict the behavior of the material under any state of stress and deformation. For example, consider condition of uniaxial strain defined by Equation 221. For this state of deformation $I_1 = \varepsilon_1$ and $\bar{I}_2' = \varepsilon_1^2/3$ , and Equation 269 results in

$$\sigma_1 = (P_0 + P_B)e^{\alpha\varepsilon_1} - P_B + \frac{\frac{2}{3}\varepsilon_1}{k_1 + \frac{k_2}{\sqrt{3}}\varepsilon_1} \tag{271}$$

$$\sigma_2 = \sigma_3 = (P_0 + P_B)e^{\alpha\varepsilon_1} - P_B - \frac{\frac{1}{3}\varepsilon_1}{k_1 + \frac{k_2}{\sqrt{3}}\varepsilon_1} \tag{272}$$

Equations 271 and 272 predict stress-strain curves that may initially be concave to the strain axis and then become concave to the stress axis as the vertical strain $\varepsilon_1$ increases. Using Equations 271 and 272 we can determine the stress path associated with the state of uniaxial strain in the $\sqrt{J_2'}$ versus $J_1/3$ space, i.e.,

$$\sqrt{J_2'} = \frac{\frac{1}{\alpha}\ln\left(\frac{\frac{J_1}{3} + P_B}{P_0 + P_B}\right)}{\sqrt{3}\,k_1 + \frac{k_2}{\alpha}\ln\left(\frac{\frac{J_1}{3} + P_B}{P_0 + P_B}\right)} \tag{273}$$

Next consider the behavior of the material under condition of uniaxial stress defined by Equation 215 (assuming that $P_0 = 0$). For this state of stress $J_1 = \sigma_1$ and $\bar{J}_2' = \sigma_1^2/3$ and Equation 270 results in

$$\varepsilon_1 = \frac{1}{3}\left[\frac{1}{\alpha}\,\ell n\left(\frac{\frac{\sigma_1}{3} + P_B}{P_B}\right)\right] + \frac{\frac{2}{3}k_1\sigma_1}{1 - \frac{k_2}{\sqrt{3}}\sigma_1} \tag{274}$$

$$\varepsilon_2 = \frac{1}{3}\left[\frac{1}{\alpha}\,\ell n\left(\frac{\frac{\sigma_1}{3} + P_B}{P_B}\right)\right] - \frac{\frac{1}{3}k_1\sigma_1}{1 - \frac{k_2}{\sqrt{3}}\sigma_1} \tag{275}$$

75. A more complicated quasi-linear elastic material model can now be constructed by assuming that the shearing response of the material is a function of both the hydrostatic and the deviatoric stresses, while the volumetric response is only a function of pressure. Accordingly, Equation 267a is still valid while instead of Equation 267b we can write

$$\sqrt{J_2'} = f_2\left(\sqrt{I_2'}\ ,\ \frac{J_1}{3}\right) \tag{276}$$

Various forms of Equation 276 can be utilized to construct a material model. A useful form for materials such as sand can be developed by using Equation 268b and assuming that the ultimate shear strength of the material is a function of $J_1/3$. For a first-order approximation we can assume that the ultimate shear strength is a linear function of hydrostatic stress; thus,

$$\frac{1}{k_2} = \tilde{k}_2 + k_3\frac{J_1}{3} \tag{277}$$

where $\tilde{k}_2$ and $k_3$ are material constants that must be determined experimentally. Utilizing Equations 277, 268a, and 268b in Equation 264 results in the following constitutive relationship

$$\sigma_{ij} = \left[ (P_0 + P_B) e^{\alpha I_1} - P_B \right] \delta_{ij}$$

$$+ \frac{\tilde{k}_2 + k_3 \left[ (P_0 + P_B) e^{\alpha I_1} - P_B \right]}{k_1 \tilde{k}_2 + k_1 k_3 \left[ (P_0 + P_B) e^{\alpha I_1} - P_B \right] + \sqrt{\overline{I}_2'}} \left( \varepsilon_{ij} - \frac{I_1}{3} \delta_{ij} \right) \tag{278}$$

It is noted that when dependence of shear strength on hydrostatic stres disappears (i.e., when $k_3 = 0$) Equation 278 reduces to Equation 269.

76. To examine the significance of the dependency of shear strength on hydrostatic stress let us consider the behavior of Equation 278 under cylindrical state of strain (Equation 255). For this state of deformation $\sqrt{\overline{I}_2'} = (\varepsilon_1 - \varepsilon_2)/\sqrt{3}$ and $I_1 = \varepsilon_1 + 2\varepsilon_2$ and we can, after arranging terms, obtain the following relationship

$$(\varepsilon_1 - \varepsilon_2) = \frac{(\sigma_1 - \sigma_2) \left\{ k_1 \tilde{k}_2 + k_1 k_3 \left[ \frac{1}{3} (\sigma_1 - \sigma_2) + \sigma_2 \right] \right\}}{\tilde{k}_2 + k_3 \left[ \frac{1}{3} (\sigma_1 - \sigma_2) + \sigma_2 \right] - \frac{1}{\sqrt{3}} (\sigma_1 - \sigma_2)} \tag{279}$$

If we consider a stress path where $\sigma_2 = \sigma_3$ is kept constant during the test (Figure 6d), Equation 279 can be used to relate stress difference $(\sigma_1 - \sigma_2)$ to strain difference $(\varepsilon_1 - \varepsilon_2)$ for a constant value of $\sigma_2$. The qualitative behavior of Equation 279 is depicted in Figure 10. It is observed from Figure 10 that the shear strength of the material increases with increasing confining stress $\sigma_2$. If $k_3$ is set to zero in Equation 279 it is noted that the dependency of shear strength on confining stress disappears.

77. The next step in developing more complicated quasi-linear elastic stress-strain relationships is to assume that volumetric strain $I_1$ is caused by both the hydrostatic and deviatoric stresses. For such material we can write

$$I_1 = g \left( \frac{J_1}{3}, \sqrt{J_2'} \right) \tag{280}$$

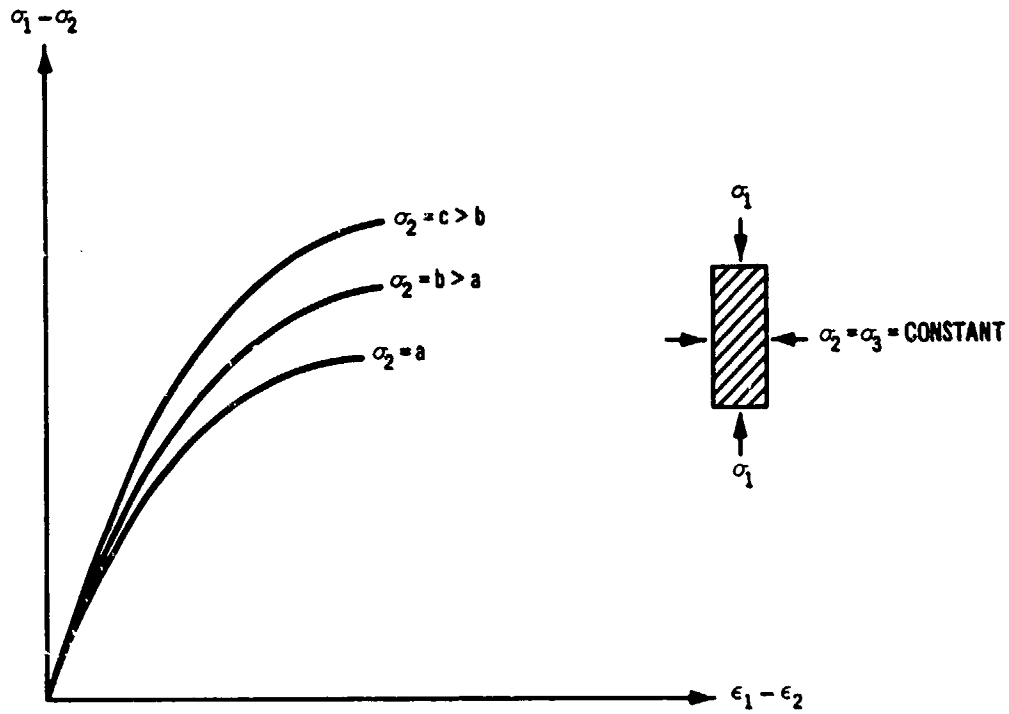



Figure 10. Response predicted by Equation 279 for cylindrical state of strain

where the function $g$ must be postulated, based on experimental results, for any material of interest. Equation 280 can be simplified further by expressing the volumetric strain $I_1$ as the sum of two components, i.e.,

$$I_1 = g_h\left(\frac{J_1}{3}\right) + g_s\left(\sqrt{J_2'}\right) \tag{281}$$

where $g_h$ = contribution due to hydrostatic stress and $g_s$ = contribution due to deviatoric stresses. For the contribution due to hydrostatic stress we can use the inverse of Equation 268a

$$g_h\left(\frac{J_1}{3}\right) = \frac{1}{\alpha} \ln\left(\frac{\frac{J_1}{3} + P_B}{P_0 + P_B}\right) \tag{282}$$

As a first-order approximation, for the contribution due to deviatoric stresses we can express $g_s$ as

$$g_s = \tilde{\alpha}\sqrt{\bar{J}_2'} \tag{283}$$

where $\tilde{\alpha}$ is a material constant that can be positive or negative depending on whether the material contracts or expands, respectively, during the application of deviatoric stresses. In view of Equations 281 through 283 the relationship for the volumetric strain (Equation 280) becomes

$$I_1 = \frac{1}{\alpha} \ln \left( \frac{\frac{J_1}{3} + P_B}{P_0 + P_B} \right) + \tilde{\alpha}\sqrt{\bar{J}_2'} \tag{284}$$

During a hydrostatic test (Figure 6a) $\bar{J}_2' = 0$ and Equation 284 reduces to Equation 282. To formulate the constitutive equation for this class of quasi-linear material we combine Equations 268b and 277 to develop an expression for $\sqrt{\bar{I}_2'}$, and then substitute this expression and Equation 284 in Equation 265. The resulting constitutive equation becomes

$$\varepsilon_{ij} = \frac{1}{3}\left[\frac{1}{\alpha} \ln \left( \frac{\frac{J_1}{3} + P_B}{P_0 + P_B} \right) + \tilde{\alpha}\sqrt{\bar{J}_2'}\right]\delta_{ij}$$

$$+ \frac{k_1\left(\tilde{k}_2 + k_3 \frac{J_1}{3}\right)}{\tilde{k}_2 + k_3 \frac{J_1}{3} - \sqrt{\bar{J}_2'}} \left(\sigma_{ij} - \frac{J_1}{3}\delta_{ij}\right) \tag{285}$$

78. Equation 285 allows for the dependency of shear strength on hydrostatic stress and the coupling of volumetric strain and deviatoric stresses. If the material constants $k_3$ and $\tilde{\alpha}$ are set to zero these cross-effects will disappear and Equation 285 will reduce to Equation 270. It should be noted that these cross-effects (in particular the coupling of volumetric strain and deviatoric stresses) are due to scalar nonlinearity (invariants) and are different from the second-order effects discussed in the development of constitutive equations for nonlinear elastic materials. To illustrate this point further, let us examine the behavior of Equation 285 under a simple shearing stress

defined by Equation 239. For this state of stress (taking $P_0$ to be zero) $J_1/3 = 0$ and $\bar{J}_2' = \tau^2$ and from Equation 285 it follows that

$$\epsilon_{ii} = I_1 = \tilde{\alpha}\tau \tag{286}$$

indicating that simple shearing stress is accompanied by volume change. However, Equation 285 indicates that there are no normal deviatoric strain components associated with the volume change, i.e., $E_{11} = E_{22} = E_{33} = 0$ . In the case of nonlinear elastic material, on the other hand, it was shown that there are normal deviatoric strains associated with volume change due to simple shearing stress (see Equation 242). Finally, it should be pointed out that for certain stress paths Equations 278 and 285 will produce inelastic stress-strain response (these stress paths will be discussed in Part V in conjunction with incremental stress-strain laws). This is a consequence of the dependency of the shearing response of the material on the hydrostatic state of stress (Equation 276). To avoid such possibility the use of Equations 278 and 285 should be restricted to stress paths where $J_1/3$ and $\bar{J}_2'$ do not decrease, i.e., they remain constant or increase.

## Green's Method

79. In order to develop the constitutive equations of elastic materials based on Green's method we first state two of the fundamental laws of mechanics:[5]

a. The first law of thermodynamics, which is a statement of the law of conservation of energy: "The work that is performed on a mechanical system by external forces plus the heat that flows into the system from the outside equals the increase of kinetic energy plus the increase of internal energy."

b. The law of kinetic energy: "The work of all the forces (internal and external) that act on a system equals the increase of kinetic energy of the system."

Mathematically, we can express the first law of thermodynamics as[5]

$$W_e + \tilde{Q} = \Delta\tilde{T} + \Delta U \tag{287}$$

where

$W_e$ = work performed on the system by external forces

$\tilde{Q}$ = heat that flows into the system

$\Delta\tilde{T}$ = increase of kinetic energy

$\Delta U$ = increase of internal energy

The law of kinetic energy can be stated as

$$W_e + W_i = \Delta\tilde{T} \tag{288}$$

where $W_i$ = the work performed by internal forces in the system. In view of Equations 287 and 288 we can write

$$W_i = \tilde{Q} - \Delta U \tag{289}$$

Since we are only dealing with mechanical energy we assume that $\tilde{Q} = 0$ and Equation 289 reduces to

$$W_i = -\Delta U \tag{290}$$

80. We will now proceed to derive the constitutive equation of Green's elastic materials. If $V$ is a material volume (region) within a deformable body and $S$ is the surface enclosing this region, and if this region undergoes an infinitesimal displacement $\tilde{\delta}u_i$ (the symbol $\tilde{\delta}$ defines a small variation), the work of external forces can be expressed as

$$\tilde{\delta}W_e = \iint_S \sigma_{ji} n_j \tilde{\delta}u_i \, dS + \iiint_V f_i \tilde{\delta}u_i \, dV \tag{291}$$

where $n_j$ = direction cosines of the outward normal to surface $S$. The first integral in Equation 291 is due to tractive forces on $S$ and can be transformed to a volume integral by using the Divergence Theorem,[5] i.e.,

$$\iint_S \sigma_{ji} n_j \tilde{\delta}u_i \, dS = \iiint_V \sigma_{ji} (\tilde{\delta}u_i)_{,j} \, dV + \iiint_V \tilde{\delta}u_i \sigma_{ji,j} \, dV \tag{292}$$

The second integral in Equation 291 is due to body forces $f_i$ acting on material in $V$ . Combining Equations 292 and 291, the work of external forces becomes

$$\tilde{\delta}W_e = \iiint_V \left[\sigma_{ji}(\tilde{\delta}u_i)_{,j} + \tilde{\delta}u_i(\sigma_{ji,j} + f_i)\right] dV \tag{293}$$

Now, let us assume that during the displacement $\tilde{\delta}u_i$ the material volume $V$ is in equilibrium and the change in kinetic energy is zero. The equations of motion (Equation 192) then take the form

$$\sigma_{ij,j} + f_i = 0 \tag{294}$$

which are referred to as the equations of equilibrium. Since the stress tensor is symmetrical, application of Equations 294 in Equation 293 results in the following expression for $\tilde{\delta}W_e$

$$\tilde{\delta}W_e = \iiint_V \sigma_{ji}(\tilde{\delta}u_i)_{,j}\, dV \tag{295}$$

Similar to Equations 149 through 151, the infinitesimal displacement gradient $(\tilde{\delta}u_i)_{,j}$ can be expressed as

$$(\tilde{\delta}u_i)_{,j} = \tilde{\delta}\varepsilon_{ij} + \tilde{\delta}\Omega_{ij} \tag{296}$$

In view of Equation 296, Equation 295 becomes

$$\tilde{\delta}W_e = \iiint_V \left(\sigma_{ji}\tilde{\delta}\varepsilon_{ij} + \sigma_{ji}\tilde{\delta}\Omega_{ij}\right) dV \tag{297}$$

Since $\sigma_{ji} = \sigma_{ij}$ (symmetry of stress tensor) and $\tilde{\delta}\Omega_{ij}$ is skew-symmetric, the expression $\sigma_{ji}\tilde{\delta}\Omega_{ij}$ is zero and Equation 297 reduces to

$$\tilde{\delta}W_e = \iiint_V \sigma_{ij}\tilde{\delta}\varepsilon_{ij}\, dV \tag{298}$$

Invoking the assumption that $\Delta\tilde{T} = 0$ during the infinitesimal displacement, and since $\tilde{Q} = 0$ , Equation 287 can be written as

$$\tilde{\delta}W_e = \tilde{\delta}U \tag{299}$$

The internal energy associated with the material volume $V$ can be expressed as

$$U = \iiint_V U_0 \, dV \tag{300}$$

where $U_0$ is the internal energy per unit volume, referred to as the internal energy density. In view of Equations 298, 299, and 300, we can write

$$\iiint_V \sigma_{ij} \tilde{\delta}\varepsilon_{ij} \, dV = \iiint_V \tilde{\delta}U_0 \, dV \tag{301}$$

which leads to

$$\tilde{\delta}U_0 = \sigma_{ij} \tilde{\delta}\varepsilon_{ij} \tag{302}$$

Since the internal energy density function $U_0$ depends on the strain components $\varepsilon_{ij}$, the variation $\tilde{\delta}U_0$ due to $\tilde{\delta}\varepsilon_{ij}$ can be expressed as

$$\tilde{\delta}U_0 = \frac{\partial U_0}{\partial \varepsilon_{ij}} \tilde{\delta}\varepsilon_{ij} \tag{303}$$

In view of Equations 302 and 303, the stress tensor $\sigma_{ij}$ takes the following form

$$\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}} \tag{304}$$

Equation 304 is referred to as the Green elastic constitutive equation.

81. For isotropic materials the strain energy function $U_0$ must be invariant and, thus, a function of strain invariants. Therefore, for isotropic materials we can write

$$\sigma_{ij} = \frac{\partial U_0\left(\bar{I}_1, \bar{I}_2, \bar{I}_3\right)}{\partial \varepsilon_{ij}} \tag{305}$$

If the material under consideration is incompressible, $I_1 = 0$ and $U_0 = U_0\left(\overline{I}_2\ ,\ \overline{I}_3\right)$. For such material, an arbitrary hydrostatic state of stress may be superimposed on the existing state of stress given by Equation 304. Using chain rule of differentiation, Equation 305 can be expressed as

$$\sigma_{ij} = \frac{\partial U_0}{\partial I_1}\frac{\partial I_1}{\partial \varepsilon_{ij}} + \frac{\partial U_0}{\partial \overline{I}_2}\frac{\partial \overline{I}_2}{\partial \varepsilon_{ij}} + \frac{\partial U_0}{\partial \overline{I}_3}\frac{\partial \overline{I}_3}{\partial \varepsilon_{ij}} \tag{306}$$

Since

$$\frac{\partial I_1}{\partial \varepsilon_{ij}} = \delta_{ij} \tag{307a}$$

$$\frac{\partial \overline{I}_2}{\partial \varepsilon_{ij}} = \varepsilon_{ij} \tag{307b}$$

$$\frac{\partial \overline{I}_3}{\partial \varepsilon_{ij}} = \varepsilon_{im}\varepsilon_{mj} \tag{307c}$$

Equation 306 can be written as

$$\sigma_{ij} = \frac{\partial U_0}{\partial I_1}\delta_{ij} + \frac{\partial U_0}{\partial \overline{I}_2}\varepsilon_{ij} + \frac{\partial U_0}{\partial \overline{I}_3}\varepsilon_{im}\varepsilon_{mj} \tag{308}$$

Comparison of Equation 308 with Equation 197 indicates that the Green and the Cauchy elastic constitutive equations have the same form. The difference between the two formulations is that the response coefficients $\phi_0$ , $\phi_1$ , and $\phi_2$ in Equation 197 are independent whereas the corresponding response coefficients $\partial U_0/\partial I_1$ , $\partial U_0/\partial \overline{I}_2$ , and $\partial U_0/\partial \overline{I}_3$ in Equation 308 are not. By differentiating the response coefficients in Equation 308 with respect to strain invariants, it follows that the following relationships exist between these coefficients

$$\frac{\partial\left(\frac{\partial U_0}{\partial \bar{I}_1}\right)}{\partial \bar{I}_2} = \frac{\partial\left(\frac{\partial U_0}{\partial \bar{I}_2}\right)}{\partial \bar{I}_1} \tag{309a}$$

$$\frac{\partial\left(\frac{\partial U_0}{\partial \bar{I}_1}\right)}{\partial \bar{I}_3} = \frac{\partial\left(\frac{\partial U_0}{\partial \bar{I}_3}\right)}{\partial \bar{I}_1} \tag{309b}$$

$$\frac{\partial\left(\frac{\partial U_0}{\partial \bar{I}_2}\right)}{\partial \bar{I}_3} = \frac{\partial\left(\frac{\partial U_0}{\partial \bar{I}_3}\right)}{\partial \bar{I}_2} \tag{309c}$$

The consequence of the above restrictions imposed on the response coefficients will be realized when we develop second- and higher-order elastic stress-strain laws using Equation 308. The Green elastic material can, therefore, be considered as a special type of Cauchy elastic material where the response coefficients are restricted by Equation 309.

82. The inverse of Equation 308 (the counterpart of Equation 198) can be determined by assuming that there exists a function $\Gamma_0$ so that

$$U_0 + \Gamma_0 = \sigma_{ij}\varepsilon_{ij} \tag{310}$$

Equation 310 holds for elastic materials where application of a positive stress increment results in a positive strain increment and vice versa. The function $\Gamma_0$ is referred to as the complementary energy density function. From Equation 310 it follows that

$$\Gamma_0 = -U_0 + \sigma_{ij}\varepsilon_{ij} \tag{311}$$

Differentiating Equation 311 with respect to $\sigma_{mn}$ yields

$$\frac{\partial \Gamma_0}{\partial \sigma_{mn}} = -\frac{\partial U_0}{\partial \sigma_{mn}} + \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial \sigma_{mn}} + \varepsilon_{ij}\frac{\partial \sigma_{ij}}{\partial \sigma_{mn}} \tag{312}$$

Since $U_0$ is a function of strain it follows that

$$\frac{\partial U_0}{\partial \sigma_{mn}} = \frac{\partial U_0}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial \sigma_{mn}} \tag{313}$$

Combining Equations 313 and 312 results in

$$\frac{\partial \Gamma_0}{\partial \sigma_{mn}} = \varepsilon_{ij}\frac{\partial \sigma_{ij}}{\partial \sigma_{mn}} + \left(\sigma_{ij} - \frac{\partial U_0}{\partial \varepsilon_{ij}}\right)\frac{\partial \varepsilon_{ij}}{\partial \sigma_{mn}} \tag{314}$$

In view of Equation 304 the second expression in Equation 314 is zero. Equation 314 then becomes

$$\frac{\partial \Gamma_0}{\partial \sigma_{mn}} = \varepsilon_{ij}\frac{\partial \sigma_{ij}}{\partial \sigma_{mn}} \tag{315}$$

Since

$$\frac{\partial \sigma_{ij}}{\partial \sigma_{mn}} = \begin{cases} 0 & i \neq m \text{ or } j \neq n \\ 1 & i = m\,,\ j = n \end{cases} \tag{316}$$

Equation 315 becomes

$$\varepsilon_{mn} = \frac{\partial \Gamma_0}{\partial \sigma_{mn}} \tag{317}$$

or

$$\varepsilon_{ij} = \frac{\partial \Gamma_0}{\partial \sigma_{ij}} \tag{318}$$

Equation 318 is the inverse form of Equation 304.

83. For isotropic materials $\Gamma_0$ is a function of stress invariants given by Equations 118 through 120, i.e.,

$$\varepsilon_{ij} = \frac{\partial \Gamma_0\left(J_1, \bar{J}_2, \bar{J}_3\right)}{\partial \sigma_{ij}} \tag{319}$$

Using chain rule of differentiation, Equation 319 can be expressed as

$$\varepsilon_{ij} = \frac{\partial \Gamma_0}{\partial J_1} \frac{\partial J_1}{\partial \sigma_{ij}} + \frac{\partial \Gamma_0}{\partial \bar{J}_2} \frac{\partial \bar{J}_2}{\partial \sigma_{ij}} + \frac{\partial \Gamma_0}{\partial \bar{J}_3} \frac{\partial \bar{J}_3}{\partial \sigma_{ij}} \tag{320}$$

Since

$$\frac{\partial J_1}{\partial \sigma_{ij}} = \delta_{ij} \tag{321a}$$

$$\frac{\partial \bar{J}_2}{\partial \sigma_{ij}} = \sigma_{ij} \tag{321b}$$

$$\frac{\partial \bar{J}_3}{\partial \sigma_{ij}} = \sigma_{im}\sigma_{mj} \tag{321c}$$

Equation 320 takes the following form

$$\varepsilon_{ij} = \frac{\partial \Gamma_0}{\partial J_1} \delta_{ij} + \frac{\partial \Gamma_0}{\partial \bar{J}_2} \sigma_{ij} + \frac{\partial \Gamma_0}{\partial \bar{J}_3} \sigma_{im}\sigma_{mj} \tag{322}$$

Equation 322 has the same form as Equation 198. The response coefficients in Equation 322, however, are not independent. It can readily be shown that relations similar to Equation 309 exist between these coefficients.

84. The complementary energy density function $\Gamma_0$ can also be expressed in terms of strain invariants by utilizing Equations 308 and 310. From Equations 308 and 310 it follows that

$$
\begin{aligned}
\Gamma_0 &= \sigma_{ij}\varepsilon_{ij} - U_0 \\
&= \left(\frac{\partial U_0}{\partial I_1}\delta_{ij} + \frac{\partial U_0}{\partial \bar{I}_2}\varepsilon_{ij} + \frac{\partial U_0}{\partial \bar{I}_3}\varepsilon_{im}\varepsilon_{mj}\right)\varepsilon_{ij} - U_0 \\
&= I_1\frac{\partial U_0}{\partial I_1} + 2\bar{I}_2\frac{\partial U_0}{\partial \bar{I}_2} + 3\bar{I}_3\frac{\partial U_0}{\partial \bar{I}_3} - U_0 \qquad (323)
\end{aligned}
$$

Since for a given material $U_0$ is known, we can use Equation 323 to express the complementary energy density function in terms of strain invariants. However, to obtain an inverse constitutive relationship (Equation 322) we need to express $\Gamma_0$ in terms of stress invariants. This can be accomplished, at least in principle, by first expressing the stress invariants in terms of strain invariants using Equation 308.[6] The resulting expressions can be inverted to obtain strain invariants in terms of stress invariants and then substituted in Equation 323 in order to express $\Gamma_0$ in terms of $J_1$, $\bar{J}_2$, and $\bar{J}_3$. We will now develop constitutive equations for various classes of elastic materials utilizing Green's method (i.e., Equations 308 and 322).

Linear elastic material

85. For linear elastic materials only terms in strain up to the first power are retained in the stress-strain relationship. It then follows from Equation 308 that the strain energy density function $U_0$ must be quadratic in strain in order for the resulting stress-strain relation to be of first order. Assuming that the initial state of stress is zero, $U_0$ takes the following form

$$U_0 = A_1\bar{I}_2 + A_2 I_1^2 \qquad (324)$$

where $A_1$ and $A_2$ are material constants. Substituting Equation 324 in Equation 308 results in

$$\sigma_{ij} = 2A_2 I_1\delta_{ij} + A_1\varepsilon_{ij} \qquad (325)$$

Equation 325 is identical with Equation 201, indicating that both Green's method and Cauchy's method result in the same stress-strain relation for linear elastic materials. In view of Equations 204 and 209 the material constants $A_2$ and $A_1$ become

$$A_2 = \frac{K}{2} - \frac{G}{3} \tag{326a}$$

$$A_1 = 2G \tag{326b}$$

The strain energy density function for linear elastic materials (Equation 324) can then be expressed in terms of shear modulus $G$ and bulk modulus $K$ as follows:

$$U_0 = 2G\left(\bar{I}_2 - \frac{1}{6} I_1^2\right) + \frac{K}{2} I_1^2 \tag{327}$$

It is noted that the expression $\left(\bar{I}_2 - I_1^2/6\right)$ is the second invariant of strain deviation tensor $\bar{I}_2'$ (Equation 164). We can therefore write

$$U_0 = 2G\bar{I}_2' + \frac{K}{2} I_1^2 \tag{328}$$

Since for linear elastic materials $J_1/3 = KI_1$ and $\sqrt{\bar{J}_2'} = 2G\sqrt{\bar{I}_2'}$ (Equation 266), the strain energy density function can also be written as

$$U_0 = \sqrt{\bar{J}_2'}\sqrt{\bar{I}_2'} + \frac{1}{2}\frac{J_1}{3} I_1 \tag{329}$$

where $\sqrt{\bar{J}_2'}\,\sqrt{\bar{I}_2'}$ and $J_1 I_1/6$ can be considered as energy due to distortion and volume change, respectively, during a deformation process. The strain energy density function can also be expressed in terms of stress invariants or various other combinations of elastic moduli and invariants. For example, the counterpart of Equation 328 becomes

$$U_0 = \frac{\bar{J}_2'}{2G} + \frac{\left(\frac{J_1}{3}\right)^2}{2K} \tag{330}$$

In view of Equations 323 and 327 the complementary and strain energy density functions for linear elastic materials are identical. For linear elastic materials we can, therefore, write

$$U_0 = \Gamma_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{331}$$

86. We can now examine the nature of the restrictions that must be placed on the linear elastic moduli due to the existence of strain energy density function $U_0$. Expanding Equation 324, we can write

$$U_0 = \left(\frac{A_1}{2} + A_2\right)\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2\right) + A_1\left(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2\right) + 2A_2\left(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{11}\varepsilon_{33}\right) \tag{332}$$

Equation 332 is quadratic in strain and can be expressed in the quadratic form

$$U_0 = \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}\varepsilon_i\varepsilon_j \tag{333}$$

where $\varepsilon_i$ and $\varepsilon_j$ denote the six independent components of the strain tensor. The matrix $c_{ij} = c_{ji}$ is expressed in terms of $A_1$ and $A_2$ and has the form

$$c_{ij} = \begin{bmatrix} \frac{A_1}{2} + A_2 & A_2 & A_2 & 0 & 0 & 0 \\ A_2 & \frac{A_1}{2} + A_2 & A_2 & 0 & 0 & 0 \\ A_2 & A_2 & \frac{A_1}{2} + A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_1 \end{bmatrix} \tag{334}$$

The strain energy density function $U_0$ is a positive quantity. According to the theory of quadratics, in order for $U_0$ to be a positive quantity all the minors of the diagonal elements of $c_{ij}$ must be positive.[5] Imposing this restriction on the diagonal minors of the matrix $c_{ij}$ leads to the following inequalities:

$$A_1 > 0 \tag{335a}$$

$$\frac{A_1}{4} + A_2 > 0 \tag{335b}$$

$$\frac{A_1}{2} + 3A_2 > 0 \tag{335c}$$

In view of Equations 326, 219, and 220, the above inequalities impose the following restrictions on the linear elastic moduli $G$ , $K$ , $E$ , and $\nu$ :

$$G > 0 \tag{336a}$$

$$K > 0 \tag{336b}$$

$$E > 0 \tag{336c}$$

$$-1 < \nu < \frac{1}{2} \tag{336d}$$

It is emphasized that negative values of Poisson's ratio $\nu$ have not been found experimentally for isotropic elastic materials.

Nonlinear elastic material

87. We will now proceed to develop constitutive equations for various classes of nonlinear elastic materials within the framework of Green's method (Equations 308 and 322). As in the development of nonlinear Cauchy elastic constitutive equations, we will start with second-order stress-strain relation where terms up to the second power are retained in the stress-strain relationship. The strain energy density function for a second-order stress-strain law, therefore, must be cubic

in strain. Assuming that the initial state of stress is zero, the most general cubic relation for $U_0$ takes the following form

$$U_0 = 2G\bar{I}_2 + \left(\frac{K}{2} - \frac{G}{3}\right)I_1^2 + A_3I_1\bar{I}_2 + A_4I_1^3 + A_5\bar{I}_3 \qquad (337)$$

where $A_3$ through $A_5$ are additional material constants associated with second power terms in the stress-strain relationship. It is noted that if third power terms in strain are neglected Equation 337 degenerates to the corresponding expression for linear elastic material (see Equations 324 and 326). Substituting Equation 337 in Equation 308 we obtain the following second-order stress-strain law

$$\sigma_{ij} = \left[\left(K - \frac{2}{3}G\right)I_1 + 3A_4I_1^2 + A_3\bar{I}_2\right]\delta_{ij} + (2G + A_3I_1)\varepsilon_{ij} + A_5\varepsilon_{im}\varepsilon_{mj} \qquad (338)$$

Equation 338 contains five material constants whereas its counterpart based on Cauchy's method (Equation 248) contains six. This reduction in material constants is a consequence of thermodynamic restrictions imposed on the response coefficients of Green elastic material (see Equation 309). It was noted that in the case of linear elastic materials both methods resulted in the same equation and their difference was not apparent. The difference between these two methods becomes more pronounced when considering higher-order nonlinear elastic and quasi-linear elastic materials.

88. Let us now consider the stress-strain relationship for third-order elastic materials. The strain energy density function for a third-order elastic material must contain strain terms up to the fourth power. Again assuming that the initial state of stress is zero, the strain energy density function for a third-order elastic material takes the following representation

$$U_0 = 2G\bar{I}_2 + \left(\frac{K}{2} - \frac{G}{3}\right)I_1^2 + A_3I_1\bar{I}_2 + A_4I_1^3 + A_5\bar{I}_3 + A_6I_1^4 + A_7\bar{I}_2^2 + A_8I_1\bar{I}_3 + A_9I_1^2\bar{I}_2 \qquad (339)$$

where $A_6$ through $A_9$ are additional material constants. It is noted that when fourth power terms in strain are neglected Equation 339 reduces to Equation 337 (strain energy function for second-order material). Substituting Equation 339 in Equation 308, we obtain the following third-order stress-strain law

$$\sigma_{ij} = \left[\left(K - \frac{2}{3} G\right)I_1 + 3A_4I_1^2 + A_3\bar{I}_2 + 4A_6I_1^3 + A_8\bar{I}_3 + 2A_9I_1\bar{I}_2\right]\delta_{ij} + \left(2G + A_3I_1 + 2A_7\bar{I}_2 + A_9I_1^2\right)\varepsilon_{ij} + (A_5 + A_8I_1)\varepsilon_{im}\varepsilon_{mj} \tag{340}$$

Equation 340 contains nine material constants. The counterpart of Equation 340 based on Cauchy's method (Equation 259), on the other hand, contains 12 material constants. Therefore, in the case of third-order stress-strain law the effect of thermodynamic restrictions is to reduce the number of material constants by three. In the next section we will consider quasi-linear elastic materials within the framework of Green's method.

Quasi-linear elastic materials

89. If the strain energy density function $U_0$ is independent of the third strain invariant $\bar{I}_3$, Equation 308 reduces to

$$\sigma_{ij} = \frac{\partial U_0}{\partial I_1}\delta_{ij} + \frac{\partial U_0}{\partial \bar{I}_2}\varepsilon_{ij} \tag{341}$$

Equation 341 is the counterpart of Equation 261 (constitutive equation of quasi-linear elastic materials based on Cauchy's method). In the case of Equation 341, however, the response coefficients are restricted by Equation 309a. From Equation 341 it follows that

$$\frac{J_1}{3} = \frac{\partial U_0}{\partial I_1} + \frac{1}{3} I_1 \frac{\partial U_0}{\partial \bar{I}_2} \tag{342a}$$

$$S_{ij} = \frac{\partial U_0}{\partial \bar{I}_2} E_{ij} \tag{342b}$$

Using Equations 124, 162, and 342, we can obtain expressions similar to Equation 263 for the response coefficients in Equation 341, i.e.,

$$\frac{\partial U_0}{\partial \overline{I}_2} = \sqrt{\frac{\overline{J}_2'}{\overline{I}_2'}} \tag{343a}$$

$$\frac{\partial U_0}{\partial I_1} = \frac{J_1}{3} - \frac{1}{3} I_1 \sqrt{\frac{\overline{J}_2'}{\overline{I}_2'}} \tag{343b}$$

The inverse of Equation 341, leading to strain-stress relationship, takes the form

$$\varepsilon_{ij} = \frac{\partial \Gamma_0}{\partial J_1} \delta_{ij} + \frac{\partial \Gamma_0}{\partial \overline{J}_2} \sigma_{ij} \tag{344}$$

90. In order to examine the nature of the restriction placed on the response coefficients by Equation 309a, we substitute Equation 343 in Equation 309a as follows:

$$\frac{\partial}{\partial \overline{I}_2}\left(\frac{J_1}{3} - \frac{1}{3} I_1 \sqrt{\frac{\overline{J}_2'}{\overline{I}_2'}}\right) = \frac{\partial}{\partial I_1}\left(\sqrt{\frac{\overline{J}_2'}{\overline{I}_2'}}\right) \tag{345}$$

Therefore, the functional forms of $J_1$ and $\sqrt{\overline{J}_2'/\overline{I}_2'}$ must satisfy the above differential equation. In the case of quasi-linear elastic material based on Cauchy's method, it was noted that the functional forms of $J_1$ and $\sqrt{\overline{J}_2'/\overline{I}_2'}$ were not restricted. If we consider a material for which $J_1$ depends on $I_1$ only (i.e., volumetric strain is caused only by hydrostatic stress), then Equation 345 reduces to

$$\frac{I_1}{3} \frac{\partial}{\partial \overline{I}_2}\left(\sqrt{\frac{\overline{J}_2'}{\overline{I}_2'}}\right) + \frac{\partial}{\partial I_1}\left(\sqrt{\frac{\overline{J}_2'}{\overline{I}_2'}}\right) = 0 \tag{346}$$

Equation 346 can be satisfied only if $\sqrt{\bar{J}_2' / \bar{I}_2'}$ is of the form

$$\sqrt{\frac{\bar{J}_2'}{\bar{I}_2'}} = \tilde{f}_2(\bar{I}_2') \tag{347}$$

Equation 347 indicates that for a quasi-linear elastic material for which $J_1$ depends only on $I_1$, the shearing response of the material is independent of hydrostatic stress. In view of Equations 341, 343, and 347, the constitutive equation for this class of material becomes

$$\sigma_{ij} = \left[\tilde{f}_1(I_1)\right]\delta_{ij} + \left[\tilde{f}_2(\bar{I}_2')\right]\left(\epsilon_{ij} - \frac{1}{3} I_1\delta_{ij}\right) \tag{348}$$

The functions $\tilde{f}_1$ and $\tilde{f}_2$ can be postulated, based on experimental evidence, for a given material. For example, Equation 269 is a special case of Equation 348 where the functions $\tilde{f}_1$ and $\tilde{f}_2$ are obtained from Equations 268a and 268b, respectively.

91. Equation 348 is the simplest form of equation for quasi-linear elastic material in that there is no coupling between the deviatoric and the volumetric responses of the material. Equations for more complicated forms of quasi-linear elastic materials can be developed by expressing the strain energy density function $U_0$ as a polynomial function of $I_1$ and $\bar{I}_2$, or by postulating mathematical expressions for $\sqrt{\bar{J}_2' / \bar{I}_2'}$ and $J_1/3$ that will satisfy Equation 345 and will include various degrees of coupling as desired.

PART V: INCREMENTAL CONSTITUTIVE EQUATIONS

92. Incremental constitutive equations are often used to describe the stress-strain behavior of materials in which the state of stress is a function of the current state of strain as well as of the stress path followed to reach that state. The general form of the constitutive equation for this class of material behavior is generally expressed as

$$\mathring{\sigma}_{ij} = \tilde{F}_{ij}(\dot{\varepsilon}_{mn}\ , \ \sigma_{rs}) \tag{349}$$

where $\tilde{F}_{ij}$ is a response function. Equation 349, which is a special case of Equation 194, expresses the components of one tensor in terms of the components of two other tensors. Therefore, in view of Equation 114, the functional form of $\tilde{F}_{ij}$ takes the following representation

$$\begin{aligned}\mathring{\sigma}_{ij} &= \eta_0\delta_{ij} + \eta_1\dot{\varepsilon}_{ij} + \eta_2\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kj} + \eta_3\sigma_{ij} \\ &+ \eta_4\sigma_{ik}\sigma_{kj} + \eta_5(\dot{\varepsilon}_{ik}\sigma_{kj} + \sigma_{ik}\dot{\varepsilon}_{kj}) \\ &+ \eta_6(\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kp}\sigma_{pj} + \sigma_{ik}\dot{\varepsilon}_{kp}\dot{\varepsilon}_{pj}) \\ &+ \eta_7(\dot{\varepsilon}_{ik\ kp}\sigma_{pj} + \sigma_{ik}\sigma_{kp}\dot{\varepsilon}_{pj}) \\ &+ \eta_8(\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kp}\sigma_{pt}\sigma_{tj} + \sigma_{ik}\sigma_{kp}\dot{\varepsilon}_{pt}\dot{\varepsilon}_{tj})\end{aligned} \tag{350}$$

where the response coefficients $\eta_0\ , \ \ldots\ , \ \eta_8$ are polynomial functions of the invariants of $\dot{\varepsilon}_{mn}$ and $\sigma_{rs}$ and the following joint invariants

$$\Pi_1 = \dot{\varepsilon}_{ab}\sigma_{ba} \tag{351a}$$

$$\Pi_2 = \dot{\varepsilon}_{ab}\sigma_{bc}\sigma_{ca} \tag{351b}$$

$$\Pi_3 = \dot{\varepsilon}_{ab}\dot{\varepsilon}_{bc}\sigma_{ca} \tag{351c}$$

$$\Pi_4 = \dot{\varepsilon}_{ab}\dot{\varepsilon}_{bc}\sigma_{cd}\sigma_{da} \tag{351d}$$

93. We can simplify Equation 350 by assuming that the materials of interest are rate independent. To eliminate time effects, Equation 350 must become homogeneous in time. This can be accomplished by eliminating all terms containing second and higher powers of $\dot{\varepsilon}_{mn}$ in Equation 350. Accordingly, the response coefficients $\eta_8$ , $\eta_6$ , and $\eta_2$ must vanish, $\eta_7$ , $\eta_5$ , and $\eta_1$ must be independent of $\dot{\varepsilon}_{mn}$ and functions of stress invariant alone, and $\eta_4$ , $\eta_3$ , and $\eta_0$ must be of degree one in $\dot{\varepsilon}_{mn}$ . Imposing the above restrictions on the response coefficients in Equation 350, we obtain

$$\dot{\sigma}_{ij} = \eta_0\delta_{ij} + \eta_1\dot{\varepsilon}_{ij} + \eta_3\sigma_{ij} + \eta_4\sigma_{ik}\sigma_{kj} + \eta_5(\dot{\varepsilon}_{ik}\sigma_{kj} + \sigma_{ik}\dot{\varepsilon}_{kj})$$
$$+ \eta_7(\dot{\varepsilon}_{ik}\sigma_{kp}\sigma_{pj} + \sigma_{ik}\sigma_{kp}\dot{\varepsilon}_{pj}) \tag{352}$$

The response coefficients $\eta_4$ , $\eta_3$ , and $\eta_0$ can now be written as

$$\eta_0 = \beta_0\dot{I}_1 + \beta_1\Pi_1 + \beta_2\Pi_2 \tag{353a}$$

$$\eta_3 = \beta_3\dot{I}_1 + \beta_4\Pi_1 + \beta_5\Pi_2 \tag{353b}$$

$$\eta_4 = \beta_6\dot{I}_1 + \beta_7\Pi_1 + \beta_8\Pi_2 \tag{353c}$$

where, similar to $\eta_7$ , $\eta_5$ , and $\eta_1$ , the response coefficients $\beta_0$ , ... , $\beta_8$ are independent of $\dot{\varepsilon}_{mn}$ and functions of stress invariants alone. Substituting Equation 353 in Equation 352, we obtain the following incremental constitutive equation for rate-independent materials

$$\dot{\sigma}_{ij} = (\beta_0 \dot{I}_1 + \beta_1 \Pi_1 + \beta_2 \Pi_2)\delta_{ij} + \eta_1 \dot{\varepsilon}_{ij}$$

$$+ (\beta_3 \dot{I}_1 + \beta_4 \Pi_1 + \beta_5 \Pi_2)\sigma_{ij}$$

$$+ (\beta_6 \dot{I}_1 + \beta_7 \Pi_1 + \beta_8 \Pi_2)\sigma_{ik}\sigma_{kj}$$

$$+ \eta_5(\dot{\varepsilon}_{ik}\sigma_{kj} + \sigma_{ik}\dot{\varepsilon}_{kj})$$

$$+ \eta_7(\dot{\varepsilon}_{ik}\sigma_{kp}\sigma_{pj} + \sigma_{ik}\sigma_{kp}\dot{\varepsilon}_{pj}) \qquad (354)$$

Equation 354 is the most general incremental constitutive relationship for rate-independent material. It contains twelve response coefficients which are polynomial functions of stress invariants. Since each term in Equation 354 contains a time derivative $d/dt$ (i.e., Equation 354 is homogeneous in time), both sides of the equation can be multiplied by $dt$ , resulting in the following differential form

$$d\sigma_{ij} = (\beta_0\, d\varepsilon_{nn} + \beta_1\, d\varepsilon_{ab}\, \sigma_{ba} + \beta_2\, d\varepsilon_{ab}\, \sigma_{bc}\sigma_{ca})\delta_{ij}$$

$$+ (\beta_3\, d\varepsilon_{nn} + \beta_4\, d\varepsilon_{ab}\, \sigma_{ba} + \beta_5\, d\varepsilon_{ab}\, \sigma_{bc}\sigma_{ca})\sigma_{ij}$$

$$+ (\beta_6\, d\varepsilon_{nn} + \beta_7\, d\varepsilon_{ab}\, \sigma_{ba} + \beta_8\, d\varepsilon_{ab}\, \sigma_{bc}\sigma_{ca})\sigma_{ik}\sigma_{kj}$$

$$+ \eta_1\, d\varepsilon_{ij} + \eta_5(d\varepsilon_{ik}\, \sigma_{kj} + \sigma_{ik}\, d\varepsilon_{kj})$$

$$+ \eta_7\, (d\varepsilon_{ik}\, \sigma_{kp}\sigma_{pj} + \sigma_{ik}\sigma_{kp}\, d\varepsilon_{pj}) \qquad (355)$$

where $d\varepsilon_{ij}$ and $d\sigma_{ij}$ are referred to as the strain increment and stress increment tensors, respectively. From Equation 355 it is apparent that incremental constitutive equations are first-order differential equations. To obtain unique solutions to these equations we need to prescribe some initial conditions. The integration of the differential equations for a given stress path and initial condition will lead

to stress-strain relationships. Various classes of incremental constitutive equations can be obtained from Equation 355 by specifying the highest degree of stress that appears on the right side of the equation. In the following sections we will develop and examine various classes of incremental constitutive equations.

## Incremental Constitutive Equation of Grade-Zero

94. If the right side of Equation 355 is independent of stress, the incremental constitutive relationship is referred to as grade-zero. In this case $\eta_7 = \eta_5 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$ , and $\eta_1$ and $\beta_0$ become constants. Thus, for grade-zero, Equation 355 reduces to

$$d\sigma_{ij} = \beta_0 \, d\varepsilon_{nn} \, \delta_{ij} + \eta_1 \, d\varepsilon_{ij} \tag{356}$$

It is noted that Equation 356 has the same form as the constitutive equation of linear elastic material (Equation 201). In order to include the linear elastic stress-strain law as a special case of Equation 355, the material constants $\beta_0$ (0, 0, 0) and $\eta_1$ (0, 0, 0) will be replaced by $K - \frac{2}{3} G$ and $2G$ , respectively. Accordingly, Equation 356 may be expressed as

$$d\sigma_{ij} = \left(K - \frac{2}{3} G\right) d\varepsilon_{nn} \, \delta_{ij} + 2G \, d\varepsilon_{ij} \tag{357}$$

Equation 357 is the constitutive equation of linear elastic material (see Equation 210) expressed in incremental form. In order to obtain a relation between stress and strain, Equation 357 must be integrated. If we adopt the condition that $\sigma_{ij} = 0$ when $\varepsilon_{ij} = 0$ , integration of Equation 357 results in the same stress-strain relations as predicted by linear elastic constitutive equation. For example, consider the condition of uniaxial strain (Equation 221) where $d\varepsilon_{nn} = d\varepsilon_1$ . For this state of deformation Equation 357 results in

$$d\sigma_1 = \left(K - \frac{2}{3} G\right) d\varepsilon_1 + 2G\ d\varepsilon_1 \tag{358a}$$

$$d\sigma_2 = \left(K - \frac{2}{3} G\right) d\varepsilon_1 \tag{358b}$$

Integrating Equation 358 and using the condition that $\sigma_1 = \sigma_2 = 0$ when $\varepsilon_1 = 0$ , we obtain

$$\sigma_1 = \left(K + \frac{4}{3} G\right)\varepsilon_1 \tag{359a}$$

$$\sigma_2 = \left(K - \frac{2}{3} G\right)\varepsilon_1 \tag{359b}$$

which are identical with the relationships predicted by linear elastic stress-strain law (Equation 222). Equation 357, which is the simplest form of incremental constitutive relationship, therefore does not manifest any nonlinear behavior.

## Incremental Constitutive Equation of Grade-One

95. If terms up to the first power of stress are retained in the right side of Equation 355, we obtain the incremental constitutive equation of grade-one. This can be achieved by allowing the response coefficients $\eta_7$ , $\beta_8$ , $\beta_7$ , $\beta_6$ , $\beta_5$ , $\beta_4$ , and $\beta_2$ to vanish, and the remaining response coefficients to take the following forms

$$\beta_0 = K - \frac{2}{3} G + \tilde{\lambda}_1 J_1 \tag{360a}$$

$$\beta_1 = \tilde{\lambda}_2 \tag{360b}$$

$$\beta_3 = \tilde{\lambda}_3 \tag{360c}$$

$$\eta_1 = 2G + \tilde{\lambda}_4 J_1 \tag{360d}$$

$$\eta_5 = \tilde{\lambda}_5 \tag{360e}$$

where $\tilde{\lambda}_1$ through $\tilde{\lambda}_5$ , K , and G are material constants that must be determined experimentally. In view of Equation 360, the incremental constitutive equation of grade-one becomes

$$d\sigma_{ij} = \left(K - \frac{2}{3} G + \tilde{\lambda}_1 J_1\right) d\varepsilon_{nn} \delta_{ij} + \tilde{\lambda}_2 d\varepsilon_{ab} \sigma_{ba} \delta_{ij}$$

$$+ \tilde{\lambda}_3 d\varepsilon_{nn} \sigma_{ij} + \left(2G + \tilde{\lambda}_4 J_1\right) d\varepsilon_{ij} + \tilde{\lambda}_5 (d\varepsilon_{ik} \sigma_{kj} + \sigma_{ik} d\varepsilon_{kj}) \quad (361)$$

Equation 361 contains seven material constants. It is noted that Equation 361 reduces to incremental constitutive equation of grade-zero (Equation 357) when the material constants $\tilde{\lambda}_1$ , $\tilde{\lambda}_2$ , $\tilde{\lambda}_3$ , $\lambda_4$ , and $\tilde{\lambda}_5$ vanish. For a given initial condition and stress path, Equation 361 can, in principle, be integrated in order to obtain stress-strain relationships. The differential equations generated from Equation 361 for various states of stress and deformation are coupled first-order equations. It is not possible, in general, to obtain closed-form solutions for these equations. Therefore, numerical integration schemes must be utilized.

96. Let us examine the behavior of Equation 361 under hydrostatic state of stress (Figure 6a). For hydrostatic state of stress, Equation 361 results in

$$d\sigma_{ii} = dJ_1 = 3\left(K - \frac{2}{3} G + \tilde{\lambda}_1 J_1\right) dI_1 + \tilde{\lambda}_2 J_1 \ dI_1$$

$$+ \tilde{\lambda}_3 J_1 \ dI_1 + (2G + \tilde{\lambda}_4 J_1) \ dI_1 + \frac{2}{3} \tilde{\lambda}_5 J_1 \ dI_1 \quad (362)$$

Equation 362 can be written as

$$dI_1 = \frac{dJ_1}{\left[3K + \left(3\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 + \tilde{\lambda}_4 + \frac{2}{3} \tilde{\lambda}_5\right) J_1\right]} \quad (363)$$

Integrating Equation 363, we obtain the following relationship between volumetric strain and pressure

$$\int_0^{I_1} dI_1 = \int_0^{J_1} \frac{dJ_1}{3K + K_1 J_1} \qquad (364a)$$

or

$$I_1 = \frac{1}{K_1} \ell n \left( \frac{3K + K_1 J_1}{3K} \right) \qquad (364b)$$

where $K_1 = 3\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 + \tilde{\lambda}_4 + 2\tilde{\lambda}_5/3$ . We can invert Equation 364b to relate pressure to volumetric strain, i.e.,

$$\frac{J_1}{3} = \frac{K}{K_1} \left( e^{K_1 I_1} - 1 \right) \qquad (365)$$

It is of interest to note that Equation 365 has the same form as Equation 268a, which was postulated to describe the pressure-volumetric strain behavior of quasi-linear elastic materials. In fact, it is noted that the ratio $K/K_1$ in Equation 365 corresponds to $P_B$ (the maximum tensile stress that the material can sustain before it breaks) in Equation 268a. It is important, however, to realize that unlike Equation 268a, Equation 365 is the outcome of a theory (i.e., the incremental constitutive equation of grade-one).

97. In order to examine the coupling of the deviatoric and volumetric responses of the incremental constitutive equation of grade-one we will examine the behavior of Equation 361 for a state of simple shearing deformation. The strain increment tensor associated with simple shearing deformation is given as

$$d\varepsilon_{ij} = \begin{bmatrix} 0 & d\varepsilon_{12} & 0 \\ d\varepsilon_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (366)$$

where it follows that $d\varepsilon_{nn} = 0$ . For this state of deformation, Equation 361 results in the following expressions for the nonvanishing components of stress increment tensor

$$d\sigma_{11} = d\sigma_{22} = 2(\tilde{\lambda}_2 + \tilde{\lambda}_5)\, d\varepsilon_{12}\, \sigma_{12} \quad (367a)$$

$$d\sigma_{33} = 2\tilde{\lambda}_2\, d\varepsilon_{12}\, \sigma_{12} \quad (367b)$$

$$d\sigma_{12} = \left[2G + \tilde{\lambda}_4(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \tilde{\lambda}_5(\sigma_{11} + \sigma_{22})\right] d\varepsilon_{12} \quad (367c)$$

Equation 367 is a set of first-order differential equations which must be integrated to relate stresses to shearing strain $\varepsilon_{12}$ . Without going through the process of integration, however, we can draw certain conclusions about the response of the material in simple shearing deformation. First, to maintain a simple shearing deformation (no volume change), normal stresses must be applied to the boundaries of the specimen. Second, since two of the normal stresses are unequal, Equation 367 predicts the occurrence of normal deviatoric stresses on the shearing planes. It is recalled that the same behavior was predicted for nonlinear elastic material (Equation 235). As was pointed out previously, the occurrence of normal deviatoric stresses on the shearing planes is a second-order effect due to tensorial nonlinearity (in this case the last term in Equation 361) and is a departure from linear theories where such nonlinearity does not exist.

## Incremental Constitutive Equation of Grade-Two

98. If we retain terms up to the second power of stress in the right side of Equation 355, the resulting incremental constitutive equation is referred to as grade-two. Thus, for incremental constitutive equation of grade-two the response coefficients $\beta_8$ , $\beta_7$ , and $\beta_5$ vanish, and the remaining response coefficients in Equation 355 take the following forms

$$\beta_0 = K - \frac{2}{3} G + \bar{\lambda}_1 J_1 + \bar{\lambda}_6 J_1^2 + \bar{\lambda}_7 \bar{J}_2 \quad (368a)$$

$$\beta_1 = \tilde{\lambda}_2 + \tilde{\lambda}_8 J_1 \tag{368b}$$

$$\beta_2 = \tilde{\lambda}_9 \tag{368c}$$

$$\beta_3 = \tilde{\lambda}_3 + \tilde{\lambda}_{10} J_1 \tag{368d}$$

$$\beta_4 = \tilde{\lambda}_{11} \tag{368e}$$

$$\beta_6 = \tilde{\lambda}_{12} \tag{368f}$$

$$\eta_1 = 2G + \tilde{\lambda}_4 J_1 + \tilde{\lambda}_{13} J_1^2 + \tilde{\lambda}_{14} \bar{J}_2 \tag{368g}$$

$$\eta_5 = \tilde{\lambda}_5 + \tilde{\lambda}_{15} J_1 \tag{368h}$$

$$\eta_7 = \tilde{\lambda}_{16} \tag{368i}$$

where $\tilde{\lambda}_1$ through $\tilde{\lambda}_{16}$, $K$, and $G$ are material constants. In view of Equation 368, the incremental constitutive equation of grade-two takes the following representation

$$\begin{aligned} d\sigma_{ij} = & \left(K - \frac{2}{3} G + \tilde{\lambda}_1 J_1 + \tilde{\lambda}_6 J_1^2 + \tilde{\lambda}_7 \bar{J}_2\right) d\varepsilon_{nn}\, \delta_{ij} \\ & + (\tilde{\lambda}_2 + \tilde{\lambda}_8 J_1)\, d\varepsilon_{ab}\, \sigma_{ba} \delta_{ij} + \tilde{\lambda}_9\, d\varepsilon_{ab}\, \sigma_{bc} \sigma_{ca} \delta_{ij} \\ & + (\tilde{\lambda}_3 + \tilde{\lambda}_{10} J_1)\, d\varepsilon_{nn}\, \sigma_{ij} + \tilde{\lambda}_{11}\, d\varepsilon_{ab}\, \sigma_{ba} \sigma_{ij} \\ & + \tilde{\lambda}_{12}\, d\varepsilon_{nn}\, \sigma_{ik} \sigma_{kj} \\ & + (2G + \tilde{\lambda}_4 J_1 + \tilde{\lambda}_{13} J_1^2 + \tilde{\lambda}_{14} \bar{J}_2)\, d\varepsilon_{ij} \\ & + (\tilde{\lambda}_5 + \tilde{\lambda}_{15} J_1)(d\varepsilon_{ik}\, \sigma_{kj} + \sigma_{ik}\, d\varepsilon_{kj}) \\ & + \tilde{\lambda}_{16} (d\varepsilon_{ik}\, \sigma_{kp} \sigma_{pj} + \sigma_{ik} \sigma_{kp}\, d\varepsilon_{pj}) \end{aligned} \tag{369}$$

Equation 369 contains 18 material constants. It is noted that Equation 369 reduces to Equation 361 (constitutive equation of grade-one) when $\tilde{\lambda}_6$ through $\tilde{\lambda}_{16}$ are set to zero. Similarly, we can develop incremental constitutive equations of grade-three (or higher grades) by retaining the third power (or higher powers) of stress in the right side of Equation 355. However, due to mathematical complexities of incremental constitutive equations of grade-one or higher, they are seldom utilized to solve actual engineering problems. For this reason, a class of incremental constitutive equations, usually referred to as variable-moduli constitutive models,[7,8] is often used for solution of many engineering problems. The variable-moduli models are relatively simple in that they do not contain second-order effects due to tensorial nonlinearity.

## Variable-Moduli Constitutive Models

99. The basic constitutive relation of the variable-moduli models is given by

$$d\sigma_{ij} = K\, d\varepsilon_{nn}\, \delta_{ij} + 2G\left(d\varepsilon_{ij} - \frac{1}{3}\, d\varepsilon_{nn}\, \delta_{ij}\right) \qquad (370)$$

It is noted that Equation 370 has the same form as the incremental constitutive equation of grade-zero (Equation 357). In the case of variable-moduli models, however, the equivalent bulk and shear moduli, $K$ and $G$, respectively, are assumed to be functions of stress invariants. Depending on the functional forms of these moduli, various classes of variable-moduli models can be constructed. Since Equation 370 does not include any second-order term, or terms involving joint invariants of stress and strain increment tensors, it can readily be integrated (for a given stress path and initial condition) to yield stress-strain relationships. Various classes of variable-moduli models are examined in the following sections.

### Constant-shear-modulus model

100. As implied, for a constant-shear-modulus model, $G$ is

constant and K is a function of stress invariants. If we further assume that volumetric strains are caused only by changes in pressure (i.e., there is no coupling between volumetric strain and deviatoric stresses), the bulk modulus K becomes a function of $J_1/3$ only. As a first-order approximation, let us assume that K is linearly related to pressure, i.e.,

$$K = K_0 + \tilde{K}_1 \frac{J_1}{3} \tag{371}$$

where $K_0$ (an initial bulk modulus) and $\tilde{K}_1$ are material constants. Substituting Equation 371 in Equation 370, we obtain the following incremental constitutive equation for a constant-G model

$$d\sigma_{ij} = \left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\varepsilon_{nn}\ \delta_{ij} + 2G\left(d\varepsilon_{ij} - \frac{1}{3}\, d\varepsilon_{nn}\ \delta_{ij}\right) \tag{372}$$

Equation 372 is also a special case of incremental constitutive equation of grade-one (Equation 361). That is, if we set the material constants $\tilde{\lambda}_5$, $\tilde{\lambda}_4$, $\tilde{\lambda}_3$, and $\tilde{\lambda}_2$ to zero, and replace K and $\tilde{\lambda}_1$ with $K_0$ and $\tilde{K}_1/3$, respectively, Equation 361 reduces to Equation 372.

101. We will now proceed to examine the behavior of Equation 372 under various states of stress and deformation. Under hydrostatic state of stress, Equation 372 reduces to

$$dI_1 = \frac{dJ_1}{3K_0 + \tilde{K}_1 J_1} \tag{373}$$

Integration of Equation 373 (with the initial conditions that $I_1 = 0$ when $J_1 = 0$) results in the following relationship between pressure and volumetric strain

$$\frac{J_1}{3} = \frac{K_0}{\tilde{K}_1}\left(e^{\tilde{K}_1 I_1} - 1\right) \tag{374}$$

Equation 374 is of the same form as Equation 365 (pressure-volumetric strain relation for incremental constitutive equation of grade-one).

102. For simple shearing deformation (Equation 366), Equation 372 predicts the same behavior as for linear elastic material (Equation 203). Under conditions of uniaxial strain, Equation 372 results in the following expressions

$$d\sigma_1 = \left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\varepsilon_1 + \frac{4}{3} G \, d\varepsilon_1 \tag{375}$$

$$d\sigma_3 = d\sigma_2 = \left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\varepsilon_1 - \frac{2}{3} G \, d\varepsilon_1 \tag{376}$$

Since there is no coupling between volumetric strain and deviatoric stresses, we can eliminate $J_1/3$ from Equations 375 and 376 by using Equation 374. Substituting Equation 374 in Equation 375 and noting that $I_1 = \varepsilon_1$ for uniaxial strain condition, we obtain

$$d\sigma_1 = K_0 e^{\tilde{K}_1 \varepsilon_1} d\varepsilon_1 + \frac{4}{3} G \, d\varepsilon_1 \tag{377}$$

Equation 377 can now be integrated to relate vertical stress $\sigma_1$ to vertical strain $\varepsilon_1$. Using the initial conditions that $\varepsilon_1 = 0$ when $\sigma_1 = 0$, integration of Equation 377 results in

$$\sigma_1 = \frac{4}{3} G\varepsilon_1 + \frac{K_0}{\tilde{K}_1}\left(e^{\tilde{K}_1 \varepsilon_1} - 1\right) \tag{378}$$

Similarly, we can obtain an expression for the radial stress $\sigma_2$

$$\sigma_2 = \frac{K_0}{\tilde{K}_1}\left(e^{\tilde{K}_1 \varepsilon_1} - 1\right) - \frac{2}{3} G\varepsilon_1 \tag{379}$$

Combining Equations 379, 378, and 374, we obtain the following relation for the stress path associated with the condition of uniaxial strain

$$\sqrt{\bar{J}_2'} = \frac{2G}{\sqrt{3}\,\tilde{K}_1} \ln\left(\frac{K_0 + \tilde{K}_1 \frac{J_1}{3}}{K_0}\right) \tag{380}$$

Equations 378 through 380 indicate that a constant-shear-modulus model, with a bulk modulus which is linearly dependent on pressure (Equation 371), can predict nonlinear stress-strain relationships under uniaxial strain condition.

103. We will next examine the behavior of Equation 372 under conditions of uniaxial stress. For uniaxial state of stress, $J_1 = \sigma_1$ and Equation 372 results in the following expression for the increment of axial stress $d\sigma_1$

$$d\sigma_1 = \left(K_0 + \tilde{K}_1 \frac{\sigma_1}{3}\right) d\varepsilon_{nn} + 2G\left(d\varepsilon_1 - \frac{1}{3}\ d\varepsilon_{nn}\right) \tag{381}$$

The increment of volumetric strain $d\varepsilon_{nn}$ can be eliminated from Equation 381 by using Equation 373 (this can be done because volumetric strain is a function of pressure alone); thus,

$$d\sigma_1 = \frac{d\sigma_1}{3} + 2G\ d\varepsilon_1 - \frac{2Gd\sigma_1}{3(3K_0 + \tilde{K}_1\sigma_1)} \tag{382}$$

Integrating Equation 382 (with initial conditions that $\sigma_1 = 0$ when $\varepsilon_1 = 0$) , we obtain

$$\varepsilon_1 = \frac{\sigma_1}{3G} + \frac{1}{3\tilde{K}_1} \ln\left(\frac{3K_0 + \tilde{K}_1\sigma_1}{3K_0}\right) \tag{383}$$

Equation 383 relates axial strain $\varepsilon_1$ to axial stress $\sigma_1$ . Similarly, we can obtain an expression for radial strain $\varepsilon_2 = \varepsilon_3$ in terms of axial stress

$$\varepsilon_2 = \frac{1}{3\tilde{K}_1} \ln\left(\frac{3K_0 + \tilde{K}_1\sigma_1}{3K_0}\right) - \frac{\sigma_1}{6G} \tag{384}$$

Again it is noted that highly nonlinear stress-strain curves can result from a constant-shear-modulus constitutive model.

104. Since the shear modulus is constant, Equation 372 can be integrated to yield a total stress-strain relation similar to that for quasi-linear elastic models. In terms of the stress and strain deviation tensors, Equation 372 results in (assuming zero initial conditions or both stresses and strains)

$$S_{ij} = 2GE_{ij} \tag{385}$$

In view of the definitions of $S_{ij}$ and $E_{ij}$ (Equations 122 and 161, respectively), Equation 385 can be written in terms of the stress and strain tensors

$$\sigma_{ij} = \frac{J_1}{3}\,\delta_{ij} + 2G\left(\varepsilon_{ij} - \frac{I_1}{3}\,\delta_{ij}\right) \tag{386}$$

It is noted that Equation 386 is a special case of quasi-linear elastic material (Equation 264). Substituting Equation 374, for $J_1/3$, into Equation 386, we obtain the following quasi-linear stress-strain relationship for the constant-shear-modulus model

$$\sigma_{ij} = \frac{K_0}{\bar{K}_1}\left(e^{\bar{K}_1 I_1} - 1\right)\delta_{ij} + 2G\left(\varepsilon_{ij} - \frac{I_1}{3}\,\delta_{ij}\right) \tag{387}$$

It is noted that Equation 387 satisfies Equation 346, indicating the existence of a strain energy function for the constant-shear-modulus model. Using Equations 343 and 387, we can derive the following expression for the strain energy function

$$U_0 = 2G\bar{I}_2 + \frac{K_0}{\bar{K}_1}\left(\frac{1}{\bar{K}_1}\,e^{\bar{K}_1 I_1} - I_1 - \frac{1}{\bar{K}_1}\right) - \frac{GI_1^2}{3} \tag{388}$$

It can easily be verified that if we substitute Equation 388 into Equation 341 we obtain Equation 387.

## Constant-Poisson's-ratio model

105. Another type of variable moduli model is the constant-Poisson's-ratio model where, by analogy to linear elastic materials, it is assumed that the ratio $G/K$ is constant. In terms of elastic Poisson's ratio $\nu$, this ratio is given by (see Equation 220)

$$\frac{G}{K} = \frac{3}{2}\frac{(1 - 2\nu)}{(1 + \nu)} = \tilde{\beta} \tag{389}$$

From Equation 389 it is obvious that $K$ and $G$ will have similar functional forms. In order to examine the consequence of this condition, let us assume that the functional form of $K$ is given by Equation 371. As was pointed out previously, Equation 371 indicates that volumetric strains are caused only by changes in pressure. In view of Equations 389 and 371 the constitutive equation takes the following form

$$d\sigma_{ij} = \left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\varepsilon_{nn}\, \delta_{ij} + 2\tilde{\beta}\left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right)\left(d\varepsilon_{ij} - \frac{1}{3}\, d\varepsilon_{nn}\delta_{ij}\right) \tag{390}$$

It can also be shown that Equation 390 is a special case of incremental constitutive equation of grade-one (Equation 361).

106. The behavior of Equation 390 under hydrostatic state of stress is identical with that of Equation 372. However, the behavior of Equation 390 under deviatoric state of stress is quite different than that of Equation 372. For example, consider a series of constant-pressure shear tests (see Figure 6c) conducted at $P_1$, $P_2$, and $P_3$, where $P_3 > P_2 > P_1$. The deviatoric response of Equation 390 for these constant-pressure tests is depicted in Figure 11a. It is observed from Figure 11a that the deviatoric response is dependent on the superimposed hydrostatic state of stress (in this case $G$ varies linearly with pressure). In the case of a constant-shear-modulus model (Equation 372), on the other hand, the deviatoric response does not depend on the superimposed hydrostatic stress. Let us now consider two

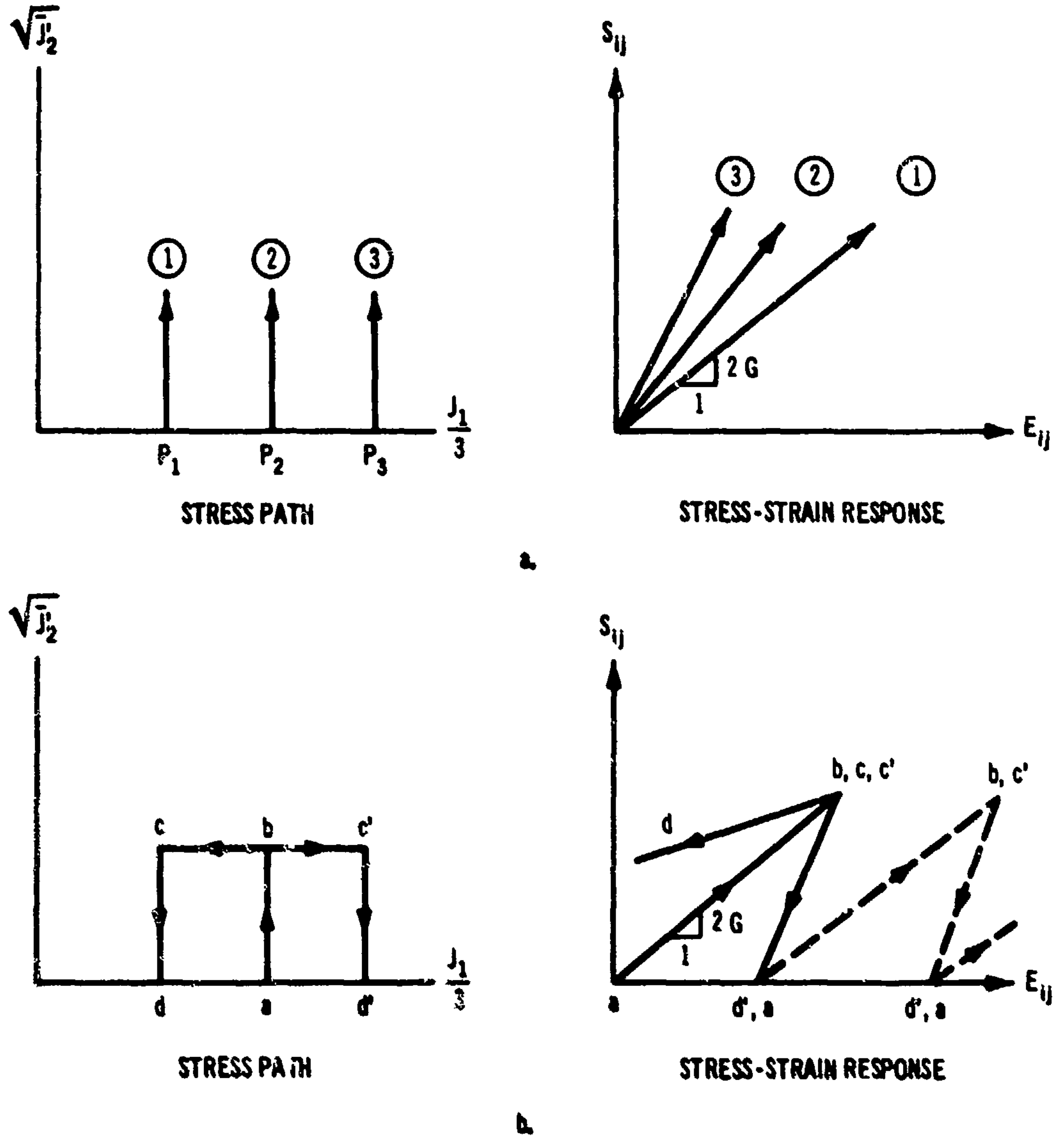



Figure 11. Behavior of constant-Poisson's-ratio variable-moduli model during constant-pressure shear test

combined constant-pressure shear and hydrostatic tests defined by stress paths abcd and abc'd' shown in Figure 11b. The deviatoric responses of Equation 390 for these two tests are also depicted in Figure 11b. For the stress path abcd it is observed that the stress-strain curve during unloading (line cd) is above the loading curve (line ab). This type of behavior results in an energy-generating loop which is contrary to the

observed behavior of real material. For the stress path abc'd' the stress-strain relation during unloading is along path c'd', which results in permanent deformation. Furthermore, if the stress cycle abc'd'a is repeated several times, the stress-strain response will result in an unrealistically excessive amount of deformation as shown by the dashed lines in Figure 11b. In order to avoid these undesirable behaviors, the use of variable-moduli models for which $G$ is a function of hydrostatic stress should be restricted to stress paths where $J_1/3$ and $\bar{J}_2'$ remain constant or increase. Variable-moduli models have been used to simulate the hysteretic behavior of earth materials during cyclic or near-cyclic conditions.[8] For this type of problem, two sets of expressions are usually specified for the moduli $K$ and $G$ : one set for loading and one set for unloading. A set of criteria or logics are also specified to determine whether the material under consideration is being loaded or unloaded so that the proper set of moduli can be used. Application of variable-moduli models for treating hysteretic effects will be discussed later.

107. Let us next consider the behavior of a constant-Poisson's-ratio model under conditions of uniaxial strain. For this state of deformation, Equation 390 results in

$$d\sigma_1 = \left(1 + \frac{4}{3}\tilde{\beta}\right)\left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\epsilon_1 \tag{391}$$

$$d\sigma_3 = d\sigma_2 = \left(1 - \frac{2}{3}\tilde{\beta}\right)\left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\epsilon_1 \tag{392}$$

We can eliminate $J_1/3$ from Equations 391 and 392 by using Equation 374. Substituting Equation 374 in Equations 391 and 392 and noting that $I_1 = \epsilon_1$ for condition of uniaxial strain, we obtain

$$d\sigma_1 = \left(1 + \frac{4}{3}\tilde{\beta}\right) K_0 e^{\tilde{K}_1 \epsilon_1} d\epsilon_1 \tag{393}$$

$$d\sigma_2 = \left(1 - \frac{2}{3}\tilde{\beta}\right) K_0 e^{\tilde{K}_1 \varepsilon_1} d\varepsilon_1 \tag{394}$$

Integrating Equations 393 and 394 and using the initial conditions that $\sigma_1 = \sigma_2 = 0$ when $\varepsilon_1 = 0$ results in

$$\sigma_1 = \frac{K_0}{\tilde{K}_1}\left(1 + \frac{4}{3}\tilde{\beta}\right)\left(e^{\tilde{K}_1 \varepsilon_1} - 1\right) \tag{395}$$

$$\sigma_2 = \frac{K_0}{\tilde{K}_1}\left(1 - \frac{2}{3}\tilde{\beta}\right)\left(e^{\tilde{K}_1 \varepsilon_1} - 1\right) \tag{396}$$

Equations 395, 396, and 374 can be combined to obtain an expression for the stress path associated with the condition of uniaxial strain, i.e.,

$$\sqrt{J_2'} = \frac{2\tilde{\beta}}{\sqrt{3}} \frac{J_1}{3} \tag{397}$$

As anticipated, the uniaxial strain stress path for a constant-Poisson's-ratio model is linear (see Equations 226 and 227). The constant-shear-modulus and constant-Poisson's-ratio models are elementary versions of variable-moduli models. More complicated forms of variable-moduli models are discussed below. These models are referred to as nonlinear variable-moduli models in that both the shearing and the volumetric responses are nonlinear and are represented by different functional forms.

Nonlinear variable-moduli models

108. More complicated, and perhaps physically more realistic, forms of variable-moduli models can be formulated by expressing K and

$G$ as separate polynomial functions of stress invariants. If we make the assumption that there is no coupling between volumetric strain and deviatoric stresses, then bulk modulus $K$ becomes a function of $J_1/3$ alone. For a first-order approximation we will adopt Equation 371 for the bulk modulus. A different functional relation can be postulated for the shear modulus $G$. As a first-order approximation, we will assume that $G$ is linearly related to the first and second invariants of stress tensor, i.e.,

$$G = G_0 + G_1 \frac{J_1}{3} + G_2 \bar{J}_2 \tag{398}$$

where $G_0$ (an initial shear modulus), $G_1$, and $G_2$ are material constants. In view of Equations 398 and 371 the constitutive relationship for this type of variable-moduli model becomes

$$d\sigma_{ij} = \left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\varepsilon_{nn}\,\delta_{ij} + 2\left(G_0 + G_1 \frac{J_1}{3} + G_2 \bar{J}_2\right)\left(d\varepsilon_{ij} - \frac{1}{3}\, d\varepsilon_{nn}\,\delta_{ij}\right) \tag{399}$$

It is noted that Equation 399 reduces to Equation 372 (constitutive equation of constant-shear-modulus model) if $G_1$ and $G_2$ are set to zero. Also, it can readily be shown that Equation 399 is a special case of incremental constitutive equation of grade-two (Equation 369).

109. The behavior of Equation 399 under hydrostatic state of stress is identical with the behavior of constant-shear-modulus and constant-Poisson's-ratio models, since the functional form of bulk modulus is the same for all these models. For deviatoric states of stress, however, the behavior of Equation 399 differs considerably from that of constant-shear-modulus and constant-Poisson's-ratio models. For example, consider a state of simple shearing deformation defined by

Equation 366. For this state of deformation, Equation 399 results in (assuming zero initial state of stress and deformation)

$$d\sigma_{11} = d\sigma_{22} = d\sigma_{33} = 0 \tag{400a}$$

$$d\sigma_{12} = 2\left(G_0 + G_2\sigma_{12}^2\right) d\varepsilon_{12} \tag{400b}$$

Integrating Equation 400b we obtain

$$\varepsilon_{12} = \frac{1}{2}\frac{1}{\sqrt{G_0G_2}} \tan^{-1}\left(\frac{\sqrt{G_0G_2}}{G_0}\,\sigma_{12}\right) \tag{401}$$

It is noted from Equations 372 and 390 that for this state of deformation the constant-shear-modulus and constant-Poisson's-ratio models predict a linear relationship between the shearing strain $\varepsilon_{12}$ and the shearing stress $\sigma_{12}$. It should also be noted that within the framework of variable-moduli models normal stresses are not required in order to maintain a state of simple shearing deformation. As was shown previously, to maintain a simple shearing deformation in the case of incremental constitutive equations of grade-one (or higher grades), normal stresses must be applied to the boundaries of the specimen (see Equation 367).

110. A more useful description for shear modulus, especially for modeling the stress-strain behavior of soil,[8] is to express $G$ in terms of $J_1/3$ and $\sqrt{\bar{J}_2'}$ rather than $\bar{J}_2$. For example, for a first-order approximation we can express $G$ as

$$G = G_0 + G_1\frac{J_1}{3} + \bar{G}_2\sqrt{\bar{J}_2'} \tag{402}$$

where $\bar{G}_2$ is a material constant. The sign of $\bar{G}_2$ will determine whether the material softens ($\bar{G}_2 < 0$) or stiffens ($\bar{G}_2 > 0$) during shear. For this description of shear modulus the constitutive equation becomes

$$d\sigma_{ij} = \left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\varepsilon_{nn}\, \delta_{ij}$$

$$+ 2\left(G_0 + G_1 \frac{J_1}{3} + \tilde{G}_2\sqrt{J_2'}\right)\left(d\varepsilon_{ij} - \frac{1}{3}\, d\varepsilon_{nn}\, \delta_{ij}\right) \qquad (403)$$

For a state of simple shearing deformation (Equation 366), Equation 403 results in (assuming zero initial state of stress and deformation)

$$d\sigma_{11} = d\sigma_{22} = d\sigma_{33} = 0 \qquad (404a)$$

$$d\sigma_{12} = 2\left(G_0 + \tilde{G}_2\sigma_{12}\right)d\varepsilon_{12} \qquad (404b)$$

Integrating Equation 404b we obtain

$$\varepsilon_{12} = \frac{1}{2}\frac{1}{\tilde{G}_2}\,\ln\left(\frac{G_0 + \tilde{G}_2\sigma_{12}}{G_0}\right) \qquad (405)$$

Equation 405 also describes a nonlinear relation between the shearing strain $\varepsilon_{12}$ and the shearing stress $\sigma_{12}$ .

111. Let us next consider the behavior of Equation 403 under cylindrical state of strain utilizing the stress path depicted in Figure 6d. For this stress path $\sqrt{J_2'} = \sqrt{3}\,(J_1/3 - \sigma_3)$ , where $\sigma_3 = \sigma_2$ is the confining stress which remains constant during the deformation process. From Equation 403 the increment of stress difference associated with cylindrical state of strain becomes

$$d\sigma_1 - d\sigma_2 = 2\left(G_0 + G_1 \frac{J_1}{3} + \tilde{G}_2\sqrt{J_2'}\right)\left(d\varepsilon_1 - d\varepsilon_2\right) \qquad (406)$$

We can eliminate $J_1/3$ from Equation 406 by using the equation for the stress path, and noting that $\sqrt{J_2'} = (\sigma_1 - \sigma_2)/\sqrt{3}$ , we obtain

$$d\varepsilon_1 - d\varepsilon_2 = \frac{d\sigma_1 - d\sigma_2}{2\left[G_0 + G_1\sigma_2 + (\sigma_1 - \sigma_2)\left(\frac{G_1}{3} + \frac{\tilde{G}_2}{\sqrt{3}}\right)\right]} \tag{407}$$

For a given value of confining stress $\sigma_2$, Equation 407 can be integrated to yield a relation between strain difference $\varepsilon_1 - \varepsilon_2$ and stress difference $\sigma_1 - \sigma_2$. Denoting the confining stress $\sigma_2$ by $\sigma_c$ and carrying on the integration, we obtain

$$\varepsilon_1 - \varepsilon_2 = \frac{1}{2\left(\frac{G_1}{3} + \frac{\tilde{G}_2}{\sqrt{3}}\right)} \times \ln\left[\frac{G_0 + G_1\sigma_c + (\sigma_1 - \sigma_2)\left(\frac{G_1}{3} + \frac{\tilde{G}_2}{\sqrt{3}}\right)}{G_0 + G_1\sigma_c}\right] \tag{408}$$

It is noted from Equation 408 that the shearing response of the material is dependent on the magnitude of the confining stress $\sigma_c$. If $G_1$ is set to zero in Equation 408, this dependency disappears (i.e., shear modulus becomes independent of hydrostatic stress, see Equation 402).

112. Next we will consider the behavior of Equation 403 under conditions of uniaxial strain. For this state of deformation, $d\varepsilon_{nn} = d\varepsilon_1$ and Equation 403 yields the following expression for the increment of vertical stress $d\sigma_1$:

$$d\sigma_1 = \left(K_0 + \tilde{K}_1 \frac{J_1}{3}\right) d\varepsilon_1 + \frac{4}{3}\left(G_0 + G_1 \frac{J_1}{3} + \tilde{G}_2 \sqrt{J_2'}\right) d\varepsilon_1 \tag{409}$$

We can eliminate $J_1/3$ and $\sqrt{J_2'}$ from Equation 409 by using Equation 374 and noting that $\sqrt{J_2'} = \sqrt{3}\,(\sigma_1 - J_1/3)/2$ ; thus,

$$\frac{d\sigma_1}{d\varepsilon_1} - \frac{2\sqrt{3}}{3}\tilde{G}_2\sigma_1 = K_0 + \frac{4}{3}G_0 + \frac{K_0}{\tilde{K}_1}\left(\tilde{K}_1 + \frac{4}{3}G_1 - \frac{2\sqrt{3}}{3}\tilde{G}_2\right)\left(e^{\tilde{K}_1\varepsilon_1} - 1\right) \tag{410}$$

It is noted that Equation 410 reduces to Equation 377 (the corresponding expression for a constant-shear-modulus model) when $\tilde{G}_2$ and $G_1$ are set to zero. Equation 410 is a first-order differential equation and has the following solution (assuming zero initial state of stress and deformation)

$$\sigma_1 = \left[\frac{\frac{K_0}{\tilde{K}_1}\left(\tilde{K}_1 + \frac{4}{3}G_1 - \frac{2\sqrt{3}}{3}\tilde{G}_2\right)}{\tilde{K}_1 - \frac{2\sqrt{3}}{3}\tilde{G}_2}\right]\left(e^{\tilde{K}_1\varepsilon_1} - e^{\frac{2\sqrt{3}}{3}\tilde{G}_2\varepsilon_1}\right)$$

$$+ \left[\frac{\frac{K_0}{\tilde{K}_1}\left(\tilde{K}_1 + \frac{4}{3}G_1 - \frac{2\sqrt{3}}{3}\tilde{G}_2\right) - \left(K_0 + \frac{4}{3}G_0\right)}{\frac{2\sqrt{3}}{3}\tilde{G}_2}\right]$$

$$\times \left(1 - e^{\frac{2\sqrt{3}}{3}\tilde{G}_2\varepsilon_1}\right) \qquad (411)$$

Again, it is noted that Equation 411 reduces to Equation 378 when $\tilde{G}_2$ and $G_1$ are set to zero. The radial stress $\sigma_2$ required to prevent radial strain can be determined by direct integration of Equation 403 (similar to the procedure followed to obtain $\sigma_1$) or by using Equations 374 and 411. From Equation 374 it follows that

$$\sigma_2 = \frac{3K_0}{2\tilde{K}_1}\left(e^{\tilde{K}_1\varepsilon_1} - 1\right) - \frac{\sigma_1}{2} \qquad (412)$$

In order to relate $\sigma_2$ to axial strain $\varepsilon_1$ we simply eliminate $\sigma_1$ from Equation 412 by using Equation 411. The uniaxial strain stress path for this model can be determined by using Equations 411 and 374. Equation 374 can be inverted to relate $I_1$ ($\varepsilon_1$ in this case) to $J_1/3$. The resulting relation can then be substituted into Equation 411 to express $\sigma_1$ in terms of $J_1/3$ which, in conjunction with the

expression $\sqrt{J_2'} = \sqrt{3}\,(\sigma_1 - J_1/3)/2$ , will result in the stress path.

113. More complicated forms of nonlinear variable-moduli models can be developed and analyzed as for the preceding models. The choice of any particular model, however, must be based on the experimental observation of the stress-strain behavior of the material of interest.

Treatment of hysteretic behavior

114. As was pointed out previously, variable-moduli models have been used to simulate the hysteretic behavior of earth materials during cyclic or near-cyclic loading conditions. To show the procedure by which the hysteretic behavior is simulated, we express the basic constitutive relation of the variable-moduli models (Equation 370) in terms of the hydrostatic and deviatoric components, i.e.,

$$\frac{dJ_1}{3} = K\, dI_1 \tag{413a}$$

$$dS_{ij} = 2G\, dE_{ij} \tag{413b}$$

where $dS_{ij}$ and $dE_{ij}$ are, respectively, the deviatoric stress and strain increment tensors. It is postulated that the basic form of Equation 413 is valid for all loading conditions (initial loading, unloading, and reloading). However, the functional forms of $K$ and $G$ change depending on whether the material under consideration is being loaded or unloaded. As depicted graphically in Figure 12, for initial loadings the response of the material is governed by

$$\frac{dJ_1}{3} = K_\ell\, dI_1 \tag{414a}$$

$$dS_{ij} = 2G_\ell\, dE_{ij} \tag{414b}$$

where $K_\ell$ and $G_\ell$ are, respectively, the bulk and shear moduli associated with initial loading. During unloading and reloading we assume that the response of the material is governed by

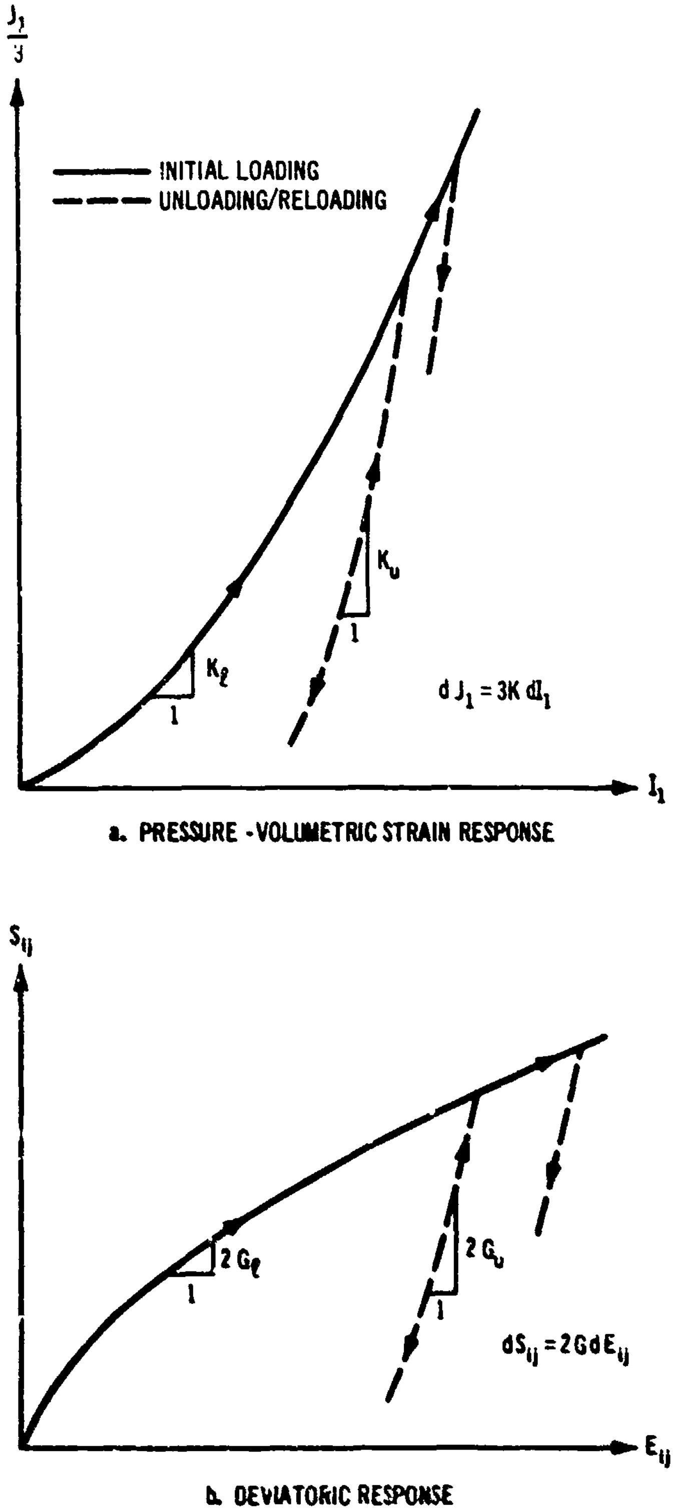



Figure 12. Treatment of hysteretic behavior using variable-moduli models

$$\frac{dJ_1}{3} = K_u \, dI_1 \tag{415a}$$

$$dS_{ij} = 2G_u \, dE_{ij} \tag{415b}$$

where $K_u$ and $G_u$ are, respectively, the bulk and shear moduli associated with unloading and reloading.

115. To complete the specification of the model we need a criterion to define the loading condition during a deformation process. We adopt as our criterion the quantity

$$dW = \sigma_{ij} \, d\varepsilon_{ij} \tag{416}$$

which defines the rate at which the stresses do work during the deformation process. According to this criterion, $dW > 0$ defines loading (initial loading or reloading) and $dw < 0$ defines unloading. The condition $dW = 0$ is referred to as neutral loading. The neutral states of loading associated with the rate of work criterion impose certain restrictions on the material constants in the constitutive equations for loading and unloading and require special considerations. The material constants must be chosen so that the loading and unloading constitutive equations become identical whenever $dW = 0$, i.e., neutral loading. This requirement must be met in order to obtain a unique solution for a boundary-value problem involving cyclic loading conditions. From Equations 414 and 415 it is apparent that variable-moduli models, in general, do not satisfy this requirement.

116. In view of the definition of deviatoric strain increment tensor we can express Equation 416 in the following form

$$dW = \left( S_{mn} + \frac{J_1}{3} \delta_{mn} \right) \left( dE_{mn} + \frac{dI_1}{3} \delta_{mn} \right) \tag{417a}$$

or

$$dW = S_{mn} \, dE_{mn} + \frac{J_1}{3} \, dI_1 \tag{417b}$$

We can eliminate $dE_{mn}$ and $dI_1$ from Equation 417b by using Equation 413; thus,

$$dW = \frac{S_{mn}\ dS_{mn}}{2G} + \frac{J_1\ dJ_1}{9K} \tag{418}$$

Since $\bar{J}_2' = 1/2\left(S_{mn}S_{mn}\right)$, it follows that

$$d\bar{J}_2' = S_{mn}\ dS_{mn} \tag{419}$$

where $d\bar{J}_2'$ is the increment of the second invariant of stress deviation tensor. Equation 418 can, therefore, be written as

$$dW = \frac{d\bar{J}_2'}{2G} + \frac{J_1\ dJ_1}{9K} \tag{420}$$

For variable-moduli models in which $G = G(\bar{J}_2')$ and $K = K(J_1)$ the rate of work can be separated into hydrostatic and deviatoric components. Equation 420 can then be used with the interpretation that $d\bar{J}_2'/2G$ is rate of work due to deviatoric stresses and $J_1\ dJ_1/9K$ is rate of work due to hydrostatic stress. Accordingly, two criteria for defining various loading conditions can be prescribed. For the deviatoric part of deformation, loading and unloading are defined according to whether $d\bar{J}_2'$ is positive or negative, respectively. For the hydrostatic part of deformation, loading and unloading are defined according to whether $dJ_1$ is positive or negative, respectively. In this manner, it is possible for the material to unload in shear while loading in pressure or vice versa. It should be pointed out that these criteria also do not satisfy the requirement of neutral loading. Because of the requirement of neutral loading, the validity of variable-moduli models for treating hysteretic effects has been questioned.[9] Hysteretic effects and permanent deformation can be treated within the framework of incremental theory of plasticity without violating the requirement of neutral loading.

## PART VI: CONSTITUTIVE EQUATIONS OF SIMPLE VISCOELASTIC MATERIALS

117. For viscoelastic materials, the state of stress is a function of both the state of strain and time rate of strain. The stress tensor can, therefore, be expressed as

$$\sigma_{ij} = \hat{F}_{ij}(\varepsilon_{mn} , \dot{\varepsilon}_{rs}) \tag{421}$$

where $\hat{F}_{ij}$ = viscoelastic response function. In view of Equation 114, the response function $\hat{F}_{ij}$ takes the following form

$$\begin{aligned}\sigma_{ij} = {} & \eta_0\delta_{ij} + \eta_1\varepsilon_{ij} + \eta_2\varepsilon_{ik}\varepsilon_{kj} + \eta_3\dot{\varepsilon}_{ij} \\ & + \eta_4\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kj} + \eta_5\left(\varepsilon_{ik}\dot{\varepsilon}_{kj} + \dot{\varepsilon}_{ik}\varepsilon_{kj}\right) \\ & + \eta_6\left(\varepsilon_{ik}\varepsilon_{kp}\dot{\varepsilon}_{pj} + \dot{\varepsilon}_{ik}\varepsilon_{kp}\varepsilon_{pj}\right) \\ & + \eta_7\left(\varepsilon_{ik}\dot{\varepsilon}_{kp}\dot{\varepsilon}_{pj} + \dot{\varepsilon}_{ik}\dot{\varepsilon}_{kp}\varepsilon_{pj}\right) \\ & + \eta_8\left(\varepsilon_{ik}\varepsilon_{kp}\dot{\varepsilon}_{pt}\dot{\varepsilon}_{tj} + \dot{\varepsilon}_{ik}\dot{\varepsilon}_{kp}\varepsilon_{pt}\varepsilon_{tj}\right)\end{aligned} \tag{422}$$

where the response coefficients $\eta_0 , \ldots , \eta_8$ are polynomial functions of the invariants of $\varepsilon_{mn}$ and $\dot{\varepsilon}_{rs}$ and the following joint invariants

$$\Pi_1 = \varepsilon_{ab}\dot{\varepsilon}_{ba} \tag{423a}$$

$$\Pi_2 = \varepsilon_{ab}\dot{\varepsilon}_{bc}\dot{\varepsilon}_{ca} \tag{423b}$$

$$\Pi_3 = \varepsilon_{ab}\varepsilon_{bc}\dot{\varepsilon}_{ca} \tag{423c}$$

$$\Pi_4 = \varepsilon_{ab}\varepsilon_{bc}\dot{\varepsilon}_{cd}\dot{\varepsilon}_{da} \tag{423d}$$

It is noted that when dependence on $\dot{\varepsilon}_{rs}$ disappears, Equation 422 reduces to the constitutive equation of Cauchy elastic material (Equation 197). Various classes of viscoelastic materials can be described

by Equation 422 by proper selection of the response coefficients $\eta_0 , \dots , \eta_8$ . We will be dealing with simple viscoelastic materials in this report.

## Kelvin-Voigt Material

118. The constitutive equation of Kelvin-Voigt material, which describes the simplest type of viscoelastic material, can be obtained from Equation 422 by allowing the coefficients $\eta_2$ , $\eta_4$ , $\eta_5$ , $\eta_6$ , $\eta_7$ , and $\eta_8$ to vanish and the remaining coefficients to take the following forms

$$\eta_0 = \lambda I_1 + \lambda_v \dot{I}_1 \tag{424a}$$

$$\eta_1 = 2G \tag{424b}$$

$$\eta_3 = 2G_v \tag{424c}$$

Accordingly, the constitutive equation of Kelvin-Voigt material becomes

$$\sigma_{ij} = \lambda I_1 \delta_{ij} + \lambda_v \dot{I}_1 \delta_{ij} + 2G\varepsilon_{ij} + 2G_v \dot{\varepsilon}_{ij} \tag{425}$$

The coefficients $\lambda_v$ and $G_v$ , analogous to the elastic constants $\lambda$ and $G$ , are the dilatational and shear viscosity coefficients, respectively. Again it is noted that if $\lambda_v$ and $G_v$ are set to zero Equation 425 reduces to the constitutive equation of linear elastic material.

119. Let us examine the behavior of Kelvin-Voigt material under uniaxial state of strain defined by Equation 221. For this state of deformation, Equation 425 results in

$$\sigma_1 = \lambda\varepsilon_1 + \lambda_v \dot{\varepsilon}_1 + 2G\varepsilon_1 + 2G_v \dot{\varepsilon}_1 \tag{426a}$$

$$\sigma_2 = \lambda\varepsilon_1 + \lambda_v \dot{\varepsilon}_1 \tag{426b}$$

where $\varepsilon_1$ is the strain in the direction of motion in a uniaxial strain configuration and $\sigma_2$ is the lateral stress required to prevent lateral

strain. Equation 426a can be written in a more compact form by collecting terms

$$\sigma_1 = (\lambda + 2G)\varepsilon_1 + (\lambda_v + 2G_v)\dot{\varepsilon}_1 \tag{427}$$

The term $\lambda + 2G$ is recognized as the constrained modulus $M$ (see Equation 225). Analogous to $M$, we denote the term $\lambda_v + 2G_v$ by $M_v$, which is the viscosity coefficient associated with the conditions of uniaxial strain. Equation 427 may now be written as

$$\sigma_1 = M\varepsilon_1 + M_v\dot{\varepsilon}_1 \tag{428}$$

Equation 428 is the counterpart of the differential equation of motion for a linear spring and a linear dashpot connected in parallel (Figure 13).

120. We will now examine the behavior of Equation 428 for an applied constant stress of magnitude $\sigma_0$. For an applied constant stress $\sigma_0$, Equation 428 becomes

$$\sigma_0 = M\varepsilon_1 + M_v\dot{\varepsilon}_1 \tag{429}$$

Integrating Equation 429 (with initial conditions that $\varepsilon_1 = 0$ at $t = 0$) we obtain

$$\varepsilon_1 = \frac{\sigma_0}{M}\left[1 - e^{-(M/M_v)t}\right] \tag{430}$$

Equation 430 describes the strain-time response in uniaxial strain condition due to an applied constant stress of magnitude $\sigma_0$ (Figure 13). In view of Equations 430 and 426b, the lateral stress-time response during application of $\sigma_0$ becomes

$$\sigma_2 = \sigma_0\left[\frac{\lambda}{M} + \left(\frac{\lambda_v}{M_v} - \frac{\lambda}{M}\right)e^{-(M/M_v)t}\right] \tag{431}$$

121. Let us next examine the behavior of Equation 428 due to application of a time-varying stress condition defined by

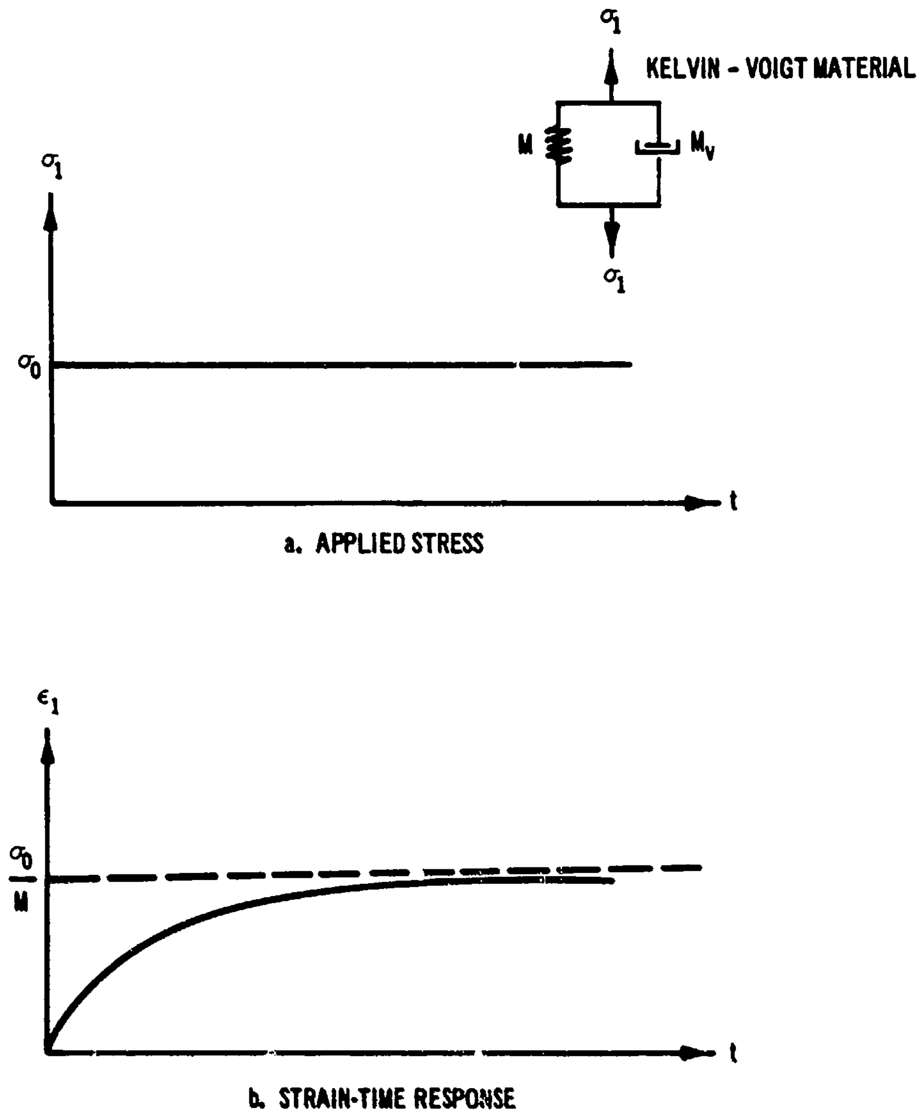



Figure 13. Behavior of Kelvin-Voigt material in uniaxial strain configuration

$$\sigma_1 = \sigma_0(1 - e^{-ct}) \tag{432}$$

where $c$ is a constant. Substituting Equation 432 in Equation 428 and integrating the resulting expression (with initial conditions that $\epsilon_1 = 0$ at $t = 0$) we obtain

$$\epsilon_1 = \frac{\sigma_0}{M}\left[1 + \frac{cM_v}{(M - cM_v)} e^{-(M/M_v)t}\right] - \frac{\sigma_0}{M - cM_v} e^{-ct} \tag{433}$$

Equation 433 describes the strain-time response in uniaxial strain condition due to application of stress-time history given by Equation 432. We can eliminate the time $t$ between Equations 432 and 433 to obtain the following strain-stress relationship

$$\varepsilon_1 = \frac{\sigma_0}{M} - \frac{\sigma_0}{M - cM_v}\left(1 - \frac{\sigma_1}{\sigma_0}\right) + \frac{c\sigma_0 M_v}{M(M - cM_v)}\left(1 - \frac{\sigma_1}{\sigma_0}\right)^{M/cM_v} \tag{434}$$

Equation 434 indicates that the stress-strain response is not unique and is dependent on the constant $c$ .

122. Analogous to nonlinear elastic material, we can construct nonlinear viscoelastic constitutive relationships by retaining some of the second-order terms in Equation 422. For example, taking $\eta_5 = \eta_6 = \eta_7 = \eta_8 = 0$ in Equation 422, a second-order viscoelastic constitutive relationship, often referred to as the nonlinear Kelvin solid, can result, i.e.,

$$\sigma_{ij} = \eta_0\delta_{ij} + \eta_1\varepsilon_{ij} + \eta_2\varepsilon_{ik}\varepsilon_{kj} + \eta_3\dot{\varepsilon}_{ij} + \eta_4\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kj} \tag{435}$$

It is noted that in Equation 435 there is no tensorial coupling between the strain and strain-rate tensors. Various classes of nonlinear Kelvin solid can be developed, similar to nonlinear elastic material, by expanding the response coefficients $\eta_0 , \ldots , \eta_4$ in terms of the invariants of the strain and strain-rate tensors.

## Maxwell Material

123. Rate-dependent constitutive relationships can also be expressed as

$$\dot{\sigma}_{ij} = \bar{F}_{ij}(\dot{\varepsilon}_{mn} , \sigma_{rs}) \tag{436}$$

Equation 436 is identical with Equation 349. The most general form of Equation 436 is given by Equation 350 and contains nine response

coefficients. If we neglect all second- and higher-order terms in Equation 350 (i.e., let $\eta_2 = \eta_4 = \eta_5 = \eta_6 = \eta_7 = \eta_8 = 0$), and assume that the remaining coefficients take the following forms

$$\eta_0 = -\alpha_m J_1 + \lambda_m \dot{I}_1 \tag{437a}$$

$$\eta_1 = 2G_m \tag{437b}$$

$$\eta_3 = -2\beta_m \tag{437c}$$

Where $\alpha_m$ , $\lambda_m$ , $G_m$ , and $\beta_m$ are material constants, we obtain

$$\dot{\sigma}_{ij} = (-\alpha_m J_1 + \lambda_m \dot{I}_1)\delta_{ij} + 2G_m \dot{\epsilon}_{ij} - 2\beta_m \sigma_{ij} \tag{438}$$

Equation 438 is the constitutive equation of Maxwell material. It contains four material constants. It is noted that if we set $\alpha_m$ and $\beta_m$ to zero and integrate Equation 438, we obtain the constitutive equation of linear elastic material (replacing $G_m$ by $G$ and $\lambda_m$ by $\lambda$).

124. Let us consider the behavior of Maxwell material under uniaxial state of stress defined by Equation 215. For this state of stress, Equation 438 results in

$$\dot{\sigma}_1 = (-\alpha_m J_1 + \lambda_m \dot{I}_1) + 2G_m \dot{\epsilon}_1 - 2\beta_m \sigma_1 \tag{439a}$$

$$0 = (-\alpha_m J_1 + \lambda_m \dot{I}_1) + 2G_m \dot{\epsilon}_2 \tag{439b}$$

where $\sigma_1$ is axial stress and $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$ are axial and lateral strain-rate components, respectively. Since for uniaxial state of stress $J_1 = \sigma_1$ and $\dot{I}_1 = \dot{\epsilon}_1 + 2\dot{\epsilon}_2$ , Equation 439 can be written in the following form

$$\dot{\sigma}_1 + (\alpha_m + 2\beta_m)\sigma_1 = (\lambda_m + 2G_m)\dot{\epsilon}_1 + 2\lambda_m \dot{\epsilon}_2 \tag{440a}$$

$$\alpha_m \sigma_1 = \lambda_m \dot{\epsilon}_1 + (2\lambda_m + 2G_m)\dot{\epsilon}_2 \tag{440b}$$

We can eliminate $\dot{\varepsilon}_2$ from Equation 440a by using Equation 440b; thus,

$$\dot{\varepsilon}_1 = \left(\frac{\lambda_m + G_m}{3G_m\lambda_m + 2G_m^2}\right)\dot{\sigma}_1 + \left(\frac{3\beta_m\lambda_m + \alpha_m G_m + 2\beta_m G_m}{3G_m\lambda_m + 2G_m^2}\right)\sigma_1 \tag{441}$$

Denoting $\left(3G_m\lambda_m + 2G_m^2\right)\Big/(\lambda_m + G_m)$ by $E_m$ (elastic Young's modulus) and $\left(3G_m\lambda_m + 2G_m^2\right)\Big/\left(3\beta_m\lambda_m + \alpha_m G_m + 2\beta_m G_m\right)$ by $\eta_m$, Equation 441 can be written as

$$\dot{\varepsilon}_1 = \frac{\dot{\sigma}_1}{E_m} + \frac{\sigma_1}{\eta_m} \tag{442}$$

Equation 442 is the counterpart of the differential equation of motion for a linear spring and a linear dashpot connected in series (Figure 14). If $\sigma_1$ is suddenly applied and then held constant $\dot{\sigma}_1 = 0$, and integration of Equation 442 results in a steady linear increase of axial strain $\varepsilon_1$ with time (Figure 14).

125. Analogous to nonlinear Kelvin solid, we may develop a nonlinear Maxwell solid by taking $\eta_5 = \eta_6 = \eta_7 = \eta_8 = 0$ in Equation 350. The constitutive equation of nonlinear Maxwell solid then becomes

$$\dot{\sigma}_{ij} = \eta_0\delta_{ij} + \eta_1\dot{\varepsilon}_{ij} + \eta_2\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kj} + \eta_3\sigma_{ij} + \eta_4\sigma_{ik}\sigma_{kj} \tag{443}$$

Equation 443 can be used to construct various classes of nonlinear Maxwell solid by expanding the response coefficients $\eta_0, \ldots, \eta_4$ in terms of the invariants of the stress and strain-rate tensors.

## Standard-Linear Material

126. Rate-dependent constitutive relationships can also be expressed as

$$\dot{\sigma}_{ij} = \hat{\delta}_{ij}(\sigma_{rs}, \dot{\varepsilon}_{mn}, \varepsilon_{pq}) \tag{444}$$

Equation 444 reduces to Equation 436 if dependence on $\varepsilon_{pq}$ disappears. The simplest form of Equation 444, which is referred to as the

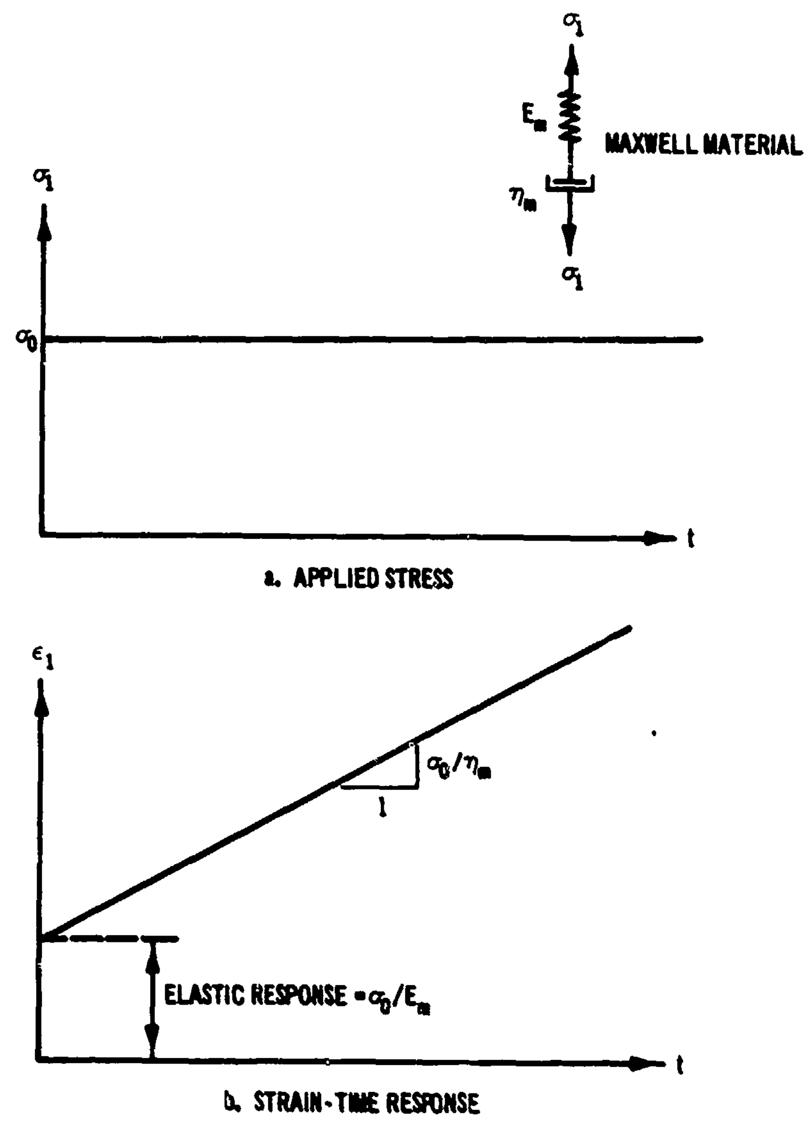



Figure 14. Behavior of Maxwell material in uniaxial stress configuration

constitutive equation of standard-linear materials, is expressed as

$$\dot{\sigma}_{ij} = \left(-\alpha_s J_1 + \bar{\lambda}_s \dot{I}_1 + \lambda_s I_1\right)\delta_{ij} + 2\bar{G}_s \dot{\epsilon}_{ij} + 2G_s \epsilon_{ij} - 2\beta_s \sigma_{ij} \qquad (445)$$

where $\alpha_s$ , $\bar{\lambda}_s$ , $\lambda_s$ , $\bar{G}_s$ , $G_s$ , and $\beta_s$ are material constants. As expected, if $G_s$ and $\lambda_s$ are set to zero, Equation 445 takes the form of the constitutive equation of Maxwell material (Equation 438).

127. Let us consider the deviatoric response of standard-linear

material. From Equation 445 it follows that

$$\dot{S}_{ij} = 2\overline{G}_s \dot{E}_{ij} + 2G_s E_{ij} - 2\beta_s S_{ij} \tag{446}$$

Equation 446 is the counterpart of the differential equation of motion for a linear spring and a Kelvin-Voigt element connected in series (Figure 15). It can be shown that the material constants $\overline{G}_s$ , $G_s$ , and $\beta_s$ and the parameters of the corresponding spring and dashpot model are related through the following relationships

$$\frac{\overline{G}_s}{\beta_s} = \frac{2\eta_s q_s}{q_s + p_s} \tag{447a}$$

$$\frac{1}{2\beta_s} = \frac{\eta_s}{q_s + p_s} \tag{447b}$$

$$\frac{G_s}{\beta_s} = \frac{2p_s q_s}{q_s + p_s} \tag{447c}$$

If the deviatoric stress (say $S_{12}$) is suddenly applied and then held constant $\dot{S}_{12} = 0$ , and integration of Equation 446 results in the following deviatoric strain-time response

$$E_{12} = \frac{S_0\beta_s}{G_s}\left[1 - \left(1 - \frac{G_s}{2\beta_s\overline{G}_s}\right)e^{-(G_s/\overline{G}_s)t}\right] \tag{448}$$

Equation 448 is depicted graphically in Figure 15. As indicated in Figure 15, the material exhibits an initial elastic response (similar to Maxwell material)

$$E_{12}(t = 0) = \frac{S_0}{2\overline{G}_s} = \frac{S_0}{2q_s} \tag{449}$$

and an asymptotic elastic behavior (similar to Kelvin-Voigt material)

$$E_{12}(t = \infty) = \frac{S_0\beta_s}{G_s} = \frac{S_0(q_s + p_s)}{2q_s p_s} \tag{450}$$

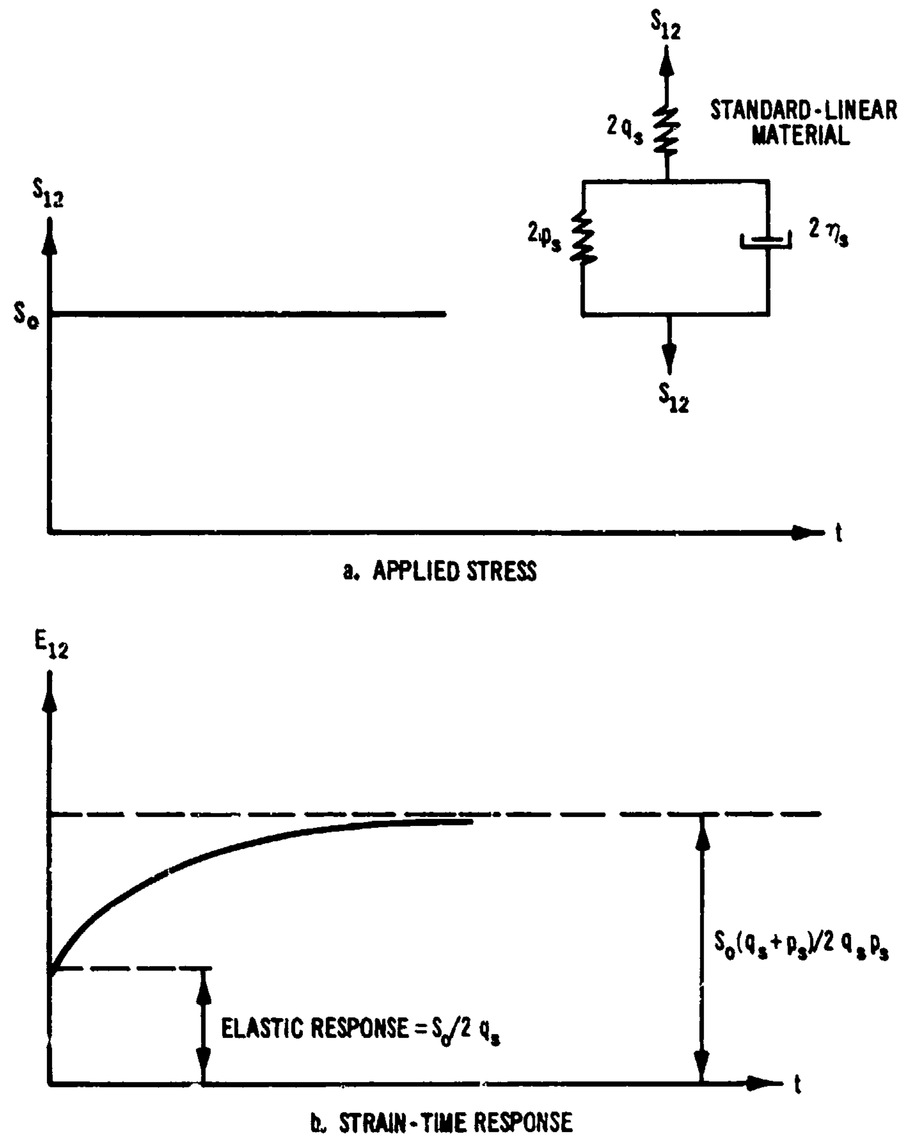



Figure 15. Behavior of standard-linear material under deviatoric state of stress

128. Similar to the nonlinear constitutive equations of Kelvin and Maxwell materials, we may develop a constitutive equation for a nonlinear standard solid containing second-order terms in stress, strain, and strain-rate tensors. This can be accomplished by combining Equations 435 (nonlinear Kelvin solid) and 443 (nonlinear Maxwell solid), i.e.,

$$\dot{\sigma}_{ij} = \eta_0 \delta_{ij} + \eta_1 \varepsilon_{ij} + \eta_2 \varepsilon_{ik} \varepsilon_{kj} + \eta_3 \dot{\varepsilon}_{ij} + \eta_4 \dot{\varepsilon}_{ik} \dot{\varepsilon}_{kj} + \eta_5 \sigma_{ij} + \eta_6 \sigma_{ik} \sigma_{kj} \tag{451}$$

Various classes of nonlinear standard solid can be developed by expanding the response coefficients $\eta_0, \ldots, \eta_6$ in terms of the invariants of the strain, strain-rate, and stress tensors.

129. Constitutive equations of viscoelastic materials can also be formulated in integral forms or series forms with differential operators.[10] Discussion of these types of constitutive equations is beyond the scope of this report.

## PART VII: CONSTITUTIVE EQUATIONS OF PLASTICITY

130. Constitutive equations of plasticity are designed to describe the stress-strain behavior of hysteretic materials. The basic assumption employed in developing these equations is that for each loading increment the corresponding strain increment can be considered as being the sum of the plastic (permanent) and elastic (recoverable) strains. Mathematically, the strain increment tensor $d\varepsilon_{ij}$ is expressed as

$$d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\varepsilon^p_{ij} \tag{452}$$

where $d\varepsilon^e_{ij}$ and $d\varepsilon^p_{ij}$ are, respectively, the elastic and plastic strain increment tensors. The elastic strain increment tensor is given in terms of incremental elastic relation

$$d\varepsilon^e_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K}\,\delta_{ij} \tag{453}$$

The elastic moduli G and K can be assumed to be constant or functions of stress invarients, as dictated by test data. However, to be consistent with path dependency of elastic materials, and to eliminate any possibility of energy generation or hysteretic behavior during elastic deformation (see Figure 11), the forms of G and K should be restricted to

$$G = G(\bar{J}'_2) \tag{454a}$$

$$K = K(J_1) \tag{454b}$$

During unloading, or during loading where the state of stress is below a specified state referred to as yield stress, the behavior of the material is defined completely by Equation 453. At the onset of yielding (when the state of stress is such that the yield stress is reached), and during subsequent loading, the material will experience both elastic and

plastic deformation and Equation 452 will govern the behavior of the material. Therefore, for a complete description of the material we need to specify the form of the plastic strain increment tensor $d\varepsilon_{ij}^{p}$. Guidelines as to how the plastic strain increment tensor can be specified were established by Drucker[11] by introducing the concept of material stability. For a stable material, the work done by a set of stress increments when applied on a specimen of the material is positive. Furthermore, if the stress increments are removed, the net work performed by them during the load-unload cycle is zero or positive. If we denote the set of stress increments by $d\sigma_{ij}$, and denote the corresponding change in the state of strain due to application of $d\sigma_{ij}$ by $d\varepsilon_{ij}$, then the first condition of stability can be stated as

$$d\sigma_{ij}\, d\varepsilon_{ij} > 0 \tag{455}$$

In view of Equation 452, the second condition of stability can be expressed as

$$d\sigma_{ij}\, d\varepsilon_{ij} - d\sigma_{ij}\, d\varepsilon_{ij}^{e} = d\sigma_{ij}\, d\varepsilon_{ij}^{p} \geq 0 \tag{456}$$

Equations 455 and 456 provide guidelines for determining the form of the plastic strain increment tensor. We also need to specify the form of the yield function $f$, which defines the limit of elastic behavior. Depending on the specification of $f$, various types of plastic constitutive equations can be established. In general, we will be concerned with ideal and work-hardening plastic materials.

## Ideal Plastic Material

131. For an ideal plastic material the yield function $f$ (or yield surface) is fixed in the principal stress space, i.e., it does not move or expand during plastic deformation. The yield surface is only a function of stress tensor, or function of invariants of stress tensor for an isotropic material. Unlimited plastic flow takes place when

$f(\sigma_{ij}) = k$ , where $k$ is a material constant defining the onset of yielding. During plastic deformation then

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \tag{457}$$

The stability condition for an ideal plastic material is specified by

$$d\sigma_{ij} \, d\varepsilon^p_{ij} = 0 \tag{458}$$

For uniaxial stress configuration, for example, Equation 458 indicates that during plastic deformation the stress remains constant while the strain increases. Since all admissible stress increments $d\sigma_{ij}$ satisfying Equation 457 must also satisfy the stability postulate given in Equation 458, it follows that

$$d\varepsilon^p_{ij} = \tilde{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \tag{459}$$

where $\tilde{\Lambda}$ is a positive scalar factor of proportionality and is dependent on the particular form of the yield function $f$ . Equation 459 is often referred to as the plastic flow rule. Inherent in Equation 459 is the normality condition which indicates that the plastic strain increment (viewed as vector) is normal to the yield surface $f$ . Another consequence of the stability postulate is the convexity condition, which requires that the yield surface $f$ must be convex in the principal stress space. In view of Equations 453 and 459, the complete expression for the strain increment tensor becomes

$$d\varepsilon_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K} \delta_{ij} + \tilde{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \tag{460}$$

Equation 460 prevails in the plastic range ($df = 0$). In the elastic range, and during unloading from a point on the yield surface ($df < 0$), the behavior of the material is governed by Equation 453.

132. In order to use Equation 460 we must determine the form of the proportionality factor $\tilde{\Lambda}$ . This can be accomplished by combining

Equations 457 and 460. From Equation 460 we can determine the stress increment tensor $d\sigma_{ij}$

$$d\sigma_{ij} = 2G\, d\varepsilon_{ij} - 2G\tilde{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} + \left(\frac{1}{3} - \frac{2G}{9K}\right) dJ_1\, \delta_{ij} \tag{461}$$

Substituting Equation 461 in Equation 457 we obtain

$$2G \frac{\partial f}{\partial \sigma_{ij}} d\varepsilon_{ij} - 2G\tilde{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \left(\frac{1}{3} - \frac{2G}{9K}\right) dJ_1 \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij} = 0 \tag{462}$$

We can eliminate $dJ_1$ from Equation 462 by using Equation 460. From Equation 460 it follows that

$$dJ_1 = 3K \left(dI_1 - \tilde{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij}\right) \tag{463}$$

In view of Equations 463 and 462, the proportionality factor $\tilde{\Lambda}$ takes the following form

$$\tilde{\Lambda} = \frac{\dfrac{\partial f}{\partial \sigma_{ij}} d\varepsilon_{ij} + \dfrac{3K - 2G}{6G} dI_1 \dfrac{\partial f}{\partial \sigma_{ij}} \delta_{ij}}{\dfrac{\partial f}{\partial \sigma_{ij}} \dfrac{\partial f}{\partial \sigma_{ij}} + \dfrac{3K - 2G}{6G} \left(\dfrac{\partial f}{\partial \sigma_{ij}} \delta_{ij}\right)^2} \tag{464}$$

It is noted that all indices in Equation 464 are dummy indices, indicating the scalar character of $\tilde{\Lambda}$ . Equations 464 and 460 can now be combined to give an expression for the strain increment tensor

$$d\varepsilon_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K} \delta_{ij} + \left[\frac{\dfrac{\partial f}{\partial \sigma_{mn}} d\varepsilon_{mn} + \dfrac{3K - 2G}{6G} dI_1 \dfrac{\partial f}{\partial \sigma_{mn}} \delta_{mn}}{\dfrac{\partial f}{\partial \sigma_{mn}} \dfrac{\partial f}{\partial \sigma_{mn}} + \dfrac{3K - 2G}{6G} \left(\dfrac{\partial f}{\partial \sigma_{mn}} \delta_{mn}\right)^2}\right] \frac{\partial f}{\partial \sigma_{ij}} \tag{465}$$

It follows from Equations 461, 463, and 464 that the stress increment tensor takes the following representation

$$d\sigma_{ij} = 2G\ dE_{ij} + K\ dI_1\ \delta_{ij} - \left[\frac{\frac{\partial f}{\partial \sigma_{mn}} d\varepsilon_{mn} + \frac{3K - 2G}{6G} dI_1 \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}}{\frac{\partial f}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{mn}} + \frac{3K - 2G}{6G}\left(\frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}\right)^2}\right]$$

$$\times \left[\left(\frac{3K - 2G}{3} \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}\right)\delta_{ij} + 2G \frac{\partial f}{\partial \sigma_{ij}}\right] \qquad (466)$$

In order to use Equations 465 and 466 we only need to define the form of the yield function $f$ for a particular material of interest. For a number of engineering materials, particularly soils, the yield function is generally expressed in terms of $J_1$ and $\sqrt{J_2'}$ i.e.,

$$f(\sigma_{ij}) = f\left(J_1\ ,\ \sqrt{J_2'}\right) = k \qquad (467)$$

For the above specification of $f$ it follows that

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial J_1} \frac{\partial J_1}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \sqrt{J_2'}} \frac{\partial \sqrt{J_2'}}{\partial \sigma_{ij}} = \frac{\partial f}{\partial J_1} \delta_{ij} + \frac{1}{2\sqrt{J_2'}} \frac{\partial f}{\partial \sqrt{J_2'}} S_{ij} \qquad (468)$$

Application of Equation 468 in Equations 465 and 466 results in

$$d\varepsilon_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K} \delta_{ij} + \left[\frac{3K\ dI_1 \frac{\partial f}{\partial J_1} + \frac{G}{\sqrt{J_2'}} \frac{\partial f}{\partial \sqrt{J_2'}} S_{mn}\ dE_{mn}}{9K\left(\frac{\partial f}{\partial J_1}\right)^2 + G\left(\frac{\partial f}{\partial \sqrt{J_2'}}\right)^2}\right]$$

$$\times \left(\frac{\partial f}{\partial J_1} \delta_{ij} + \frac{1}{2\sqrt{J_2'}} \frac{\partial f}{\partial \sqrt{J_2'}} S_{ij}\right) \qquad (469)$$

and

$$d\sigma_{ij} = 2G\, dE_{ij} + K\, dI_1\, \delta_{ij} - \left[ \frac{3K\, dI_1 \dfrac{\partial f}{\partial J_1} + \dfrac{G}{\sqrt{J_2'}} \dfrac{\partial f}{\partial \sqrt{J_2'}} S_{mn}\, dE_{mn}}{9K \left( \dfrac{\partial f}{\partial J_1} \right)^2 + G \left( \dfrac{\partial f}{\partial \sqrt{J_2'}} \right)^2} \right]$$

$$\times \left( 3K \frac{\partial f}{\partial J_1} \delta_{ij} + \frac{G}{\sqrt{J_2'}} \frac{\partial f}{\partial \sqrt{J_2'}} S_{ij} \right) \qquad (470)$$

Equations 469 and 470 are, therefore, special cases of Equations 465 and 466, respectively, where the yield function $f$ is restricted by Equation 467. In the next section we will discuss the procedure by which these equations can be utilized for specific yield functions.

Prandtl-Reuss material

133. Prandtl-Reuss material is the most widely used, and perhaps the simplest, ideal elastic-plastic material. The yield condition associated with the Prandtl-Reuss material is the well-known Von Mises criterion given by

$$f = \sqrt{J_2'} = k \qquad (471)$$

Equation 471 describes a right-circular cylinder in the principal stress space with its central axis the line of hydrostatic stress as shown in Figure 16. When the state of stress is such that Equation 471 is satisfied, the material would flow plastically, undergoing plastic as well as elastic strains. When the stresses are less than those satisfying Equation 471, the material will undergo elastic strains only.

134. In order to obtain the constitutive equation of Prandtl-Reuss material we simply substitute Equation 471, for the yield function $f$, into Equation 469. Completing the substitution, and considering the fact that during plastic deformation $\sqrt{J_2'} = k$, we obtain

$$d\varepsilon_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K} \delta_{ij} + \frac{S_{mn}\, dE_{mn}}{2k^2} S_{ij} \qquad (472)$$

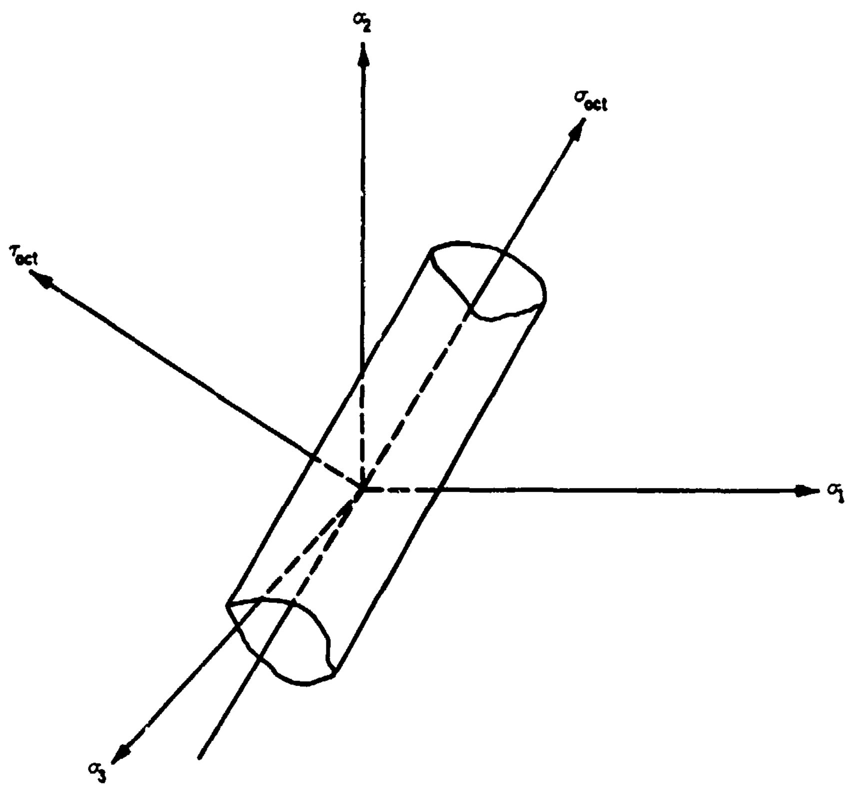



Figure 16. Von Mises yield surface in principal stress space

Similarly, substituting Equation 471 in Equation 470 we obtain the following expression for stress increment tensor

$$d\sigma_{ij} = 2G\ dE_{ij} + K\ dI_1\ \delta_{ij} - \frac{GS_{mn}\ dE_{mn}}{k^2} S_{ij} \tag{473}$$

The quantity $S_{mn}\ dE_{mn}$ in Equations 472 and 473 is recognized as the rate of work due to distortion. Expanding this quantity with respect to the plastic and elastic components we obtain

$$S_{mn}\ dE_{mn} = S_{mn}\left(dE^e_{mn} + dE^p_{mn}\right) \tag{474}$$

Since $dE^e_{mn} = dS_{mn}/2G$ (see Equation 413b), Equation 474 becomes

$$S_{mn}\ dE_{mn} = \frac{S_{mn}\ dS_{mn}}{2G} + S_{mn}\ dE^p_{mn} \tag{475}$$

The quantity $S_{mn}\ dS_{mn}$ is the increment of the second invariant of

stress deviation tensor (see Equation 419) and is zero for the Von Mises yield criterion. Equation 475 reduces to

$$S_{mn}\ dE_{mn} = S_{mn}\ dE^p_{mn} \tag{476}$$

indicating that in the plastic range the rate of distortional work is only due to plastic deformation. Also, from Equations 472 and 473 it follows that

$$d\varepsilon_{ii} = \frac{dJ_1}{3K} = d\varepsilon^e_{ii} \tag{477}$$

In view of Equation 452, Equation 477 indicates that

$$d\varepsilon^p_{ii} = 0 \tag{478}$$

That is, no plastic volume change can occur in the plastic range for Prandtl-Reuss material.

135. We can now summarize the Prandtl-Reuss equation in the following manner. During elastic loading $\left(\sqrt{J_2'} < k\right)$ and during unloading $\left((\partial f/\partial \sigma_{ij}) d\sigma_{ij} < 0\right)$, the elastic constitutive equation (Equation 453) prevails. In the plastic range $\left(\sqrt{J_2'} = k \text{ and } (\partial f/\partial \sigma_{ij}) d\sigma_{ij} = 0\right)$, Equation 472 (or Equation 473) governs the behavior of the material. The Prandtl-Reuss constitutive equation can then be expressed as

$$\begin{cases} d\sigma_{ij} = 2G\ dE_{ij} + K\ dI_1\ \delta_{ij} \\ \quad \text{when } \sqrt{J_2'} < k \text{ or } \dfrac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 & \text{(479a)} \\ d\sigma_{ij} = 2G\ dE_{ij} + K\ dI_1\ \delta_{ij} - \dfrac{GS_{mn}\ dE_{mn}}{k^2} S_{ij} \\ \quad \text{when } \sqrt{J_2'} = k \text{ and } \dfrac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 & \text{(479b)} \end{cases}$$

136. In order to demonstrate the application of Equation 479, we will examine the behavior of Prandtl-Reuss material under conditions of

uniaxial strain. For uniaxial strain conditions the strain increment and strain deviation increment tensors are given as

$$d\varepsilon_{ij} = \begin{bmatrix} d\varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{480a}$$

$$dE_{ij} = \begin{bmatrix} \frac{2}{3}\, d\varepsilon_1 & 0 & 0 \\ 0 & -\frac{1}{3}\, d\varepsilon_1 & 0 \\ 0 & 0 & -\frac{1}{3}\, d\varepsilon_1 \end{bmatrix} \tag{480b}$$

In the elastic range the behavior of the material is governed by Equation 479a.

$$d\sigma_1 = \left(K + \frac{4}{3}\, G\right) d\varepsilon_1 = M\, d\varepsilon_1 = \left(\frac{3K + 4G}{9K}\right) dJ_1 \tag{481a}$$

$$d\sigma_1 - d\sigma_2 = d\sigma_1 - d\sigma_3 = 2G\, d\varepsilon_1 = \frac{2G}{3K}\, dJ_1 \tag{481b}$$

For virgin loading in the elastic range, Equation 481 governs the behavior of the material. It should be noted that if the initial state of stress and strain is zero, for virgin loading, Equation 481 can be used in terms of total rather than incremental quantities. In the uniaxial strain test

$$\sqrt{J_2'} = \frac{1}{\sqrt{3}}\, (\sigma_1 - \sigma_2) \tag{482}$$

Thus the material will yield when

$$\frac{1}{\sqrt{3}}\, (\sigma_1 - \sigma_2) = k \tag{483}$$

In view of Equations 481 and 483, the value of vertical stress $\sigma_1$ at yield becomes

$$\sigma_1 = \frac{\sqrt{3}\,(3K + 4G)}{6G}\,k = \frac{\sqrt{3}\,M}{2G}\,k \tag{484}$$

Thus, when $\sigma_1$ reaches the value given by Equation 484, the material yields and continued application of vertical stress causes the material to move along the yield surface, undergoing plastic as well as elastic strains. In the plastic range Equation 479a no longer applies and recourse to Equation 479b is necessary. According to Equation 479b, the deviator stress increment $dS_1$ in the plastic range is given by

$$dS_1 = 2G\,dE_1 - \frac{GS_{mn}\,dE_{mn}}{k^2}\,S_1 \tag{485}$$

The rate of work $S_{mn}\,dE_{mn}$ for conditions of uniaxial strain reduces to

$$S_{mn}\,dE_{mn} = S_1\,dE_1 + 2S_2\,dE_2 \tag{486}$$

Utilizing the fact that $S_{ii} = dE_{ii} = 0$ Equation 486 reduces to

$$S_{mn}\,dE_{mn} = \frac{3}{2}\,S_1\,dE_1 \tag{487}$$

In view of Equations 480b and 487, Equation 485 becomes

$$dS_1 = \frac{4G}{3}\,d\varepsilon_1 - \frac{GS_1^2}{k^2}\,d\varepsilon_1 \tag{488}$$

In the plastic range $k^2 = \bar{J}_2' = \frac{3}{4}\,S_1^2$ and Equation 488 reduces to

$$dS_1 = 0 \tag{489}$$

Since $dS_{ii} = 0$, Equation 489 indicates that $dS_2 = 0$ also, and

$$d\sigma_1 = dS_1 + dJ_1/3 = dJ_1/3 \tag{490a}$$

$$d\sigma_2 = dS_2 + dJ_1/3 = dJ_1/3 \tag{490b}$$

Equation 490 indicates that the material behaves as though it were a fluid once it has reached its limiting shear resistance. From Equations 479b and 480a, it follows that

$$dJ_1/3 = K\, d\varepsilon_1 \tag{491}$$

Substituting Equation 491 into Equation 490a, the vertical stress-strain increment relation in the plastic range becomes

$$d\sigma_1 = K\, d\varepsilon_1 \tag{492}$$

Thus, the loading slope of the $\sigma_1$ versus $\varepsilon_1$ curve breaks, or softens, when yielding occurs and becomes equal to the bulk modulus. Accordingly, the loading slopes of the principal stress difference-pressure curve and the principal stress difference-strain difference curve become zero. Since $d\varepsilon_{kk}^p = 0$, the slope of the pressure-volumetric strain curve remains constant. Once the material unloads, it behaves as a linear elastic solid again, satisfying Equation 481. If unloading is continued until the lower yield surface corresponding to

$$\frac{1}{\sqrt{3}}(\sigma_1 - \sigma_2) = -k \tag{493}$$

is reached, the material flows plastically again and Equation 479b governs the behavior of the material. The foregoing analyses are depicted schematically in Figure 17. From Figure 17 it can readily be seen that for a Prandtl-Reuss material, the vertical stress-strain curve associated with uniaxial strain configuration would break or soften when yielding occurs and would remain concave to the strain axis with continued application of vertical stress.

137. Let us next examine the behavior of Prandtl-Reuss material under a plane stress condition defined by

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{494}$$

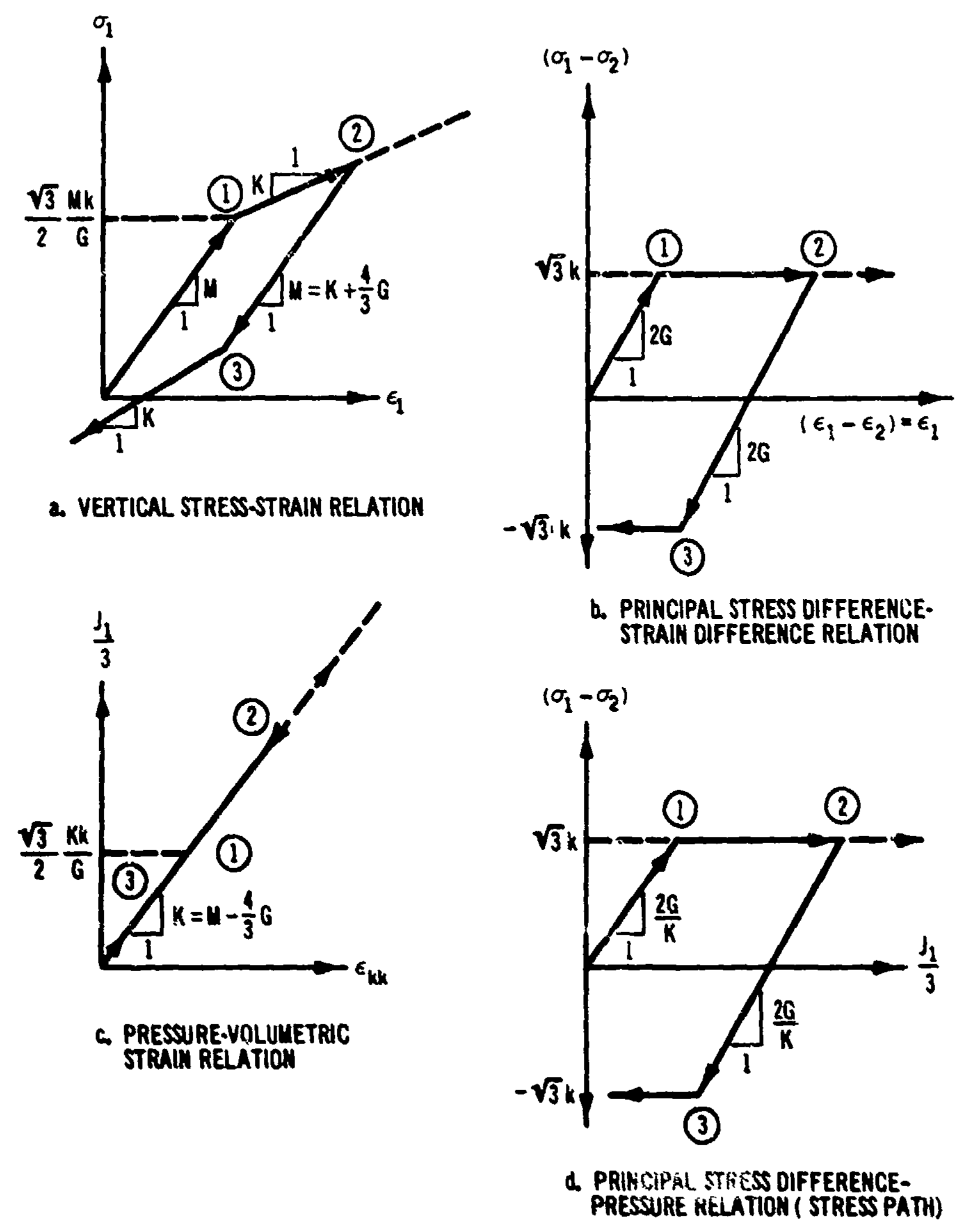



Figure 17. Behavior of Prandtl-Reuss material under conditions of uniaxial strain

For this state of stress

$$\bar{J}_2' = \frac{1}{3}\left(\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3\right) \tag{495}$$

Thus, the material will yield when

$$\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 = 3k^2 \tag{496}$$

Equation 496 describes an ellipse in the $\sigma_1$ , $\sigma_3$ coordinate system (Figure 18). We will consider a stress path where $\sigma_3$ is held constant at k while $\sigma_1$ is increased. At the start of the test (assume a compression test), point A in Figure 18, $\sigma_1 = 0$ According to Equation 496 the material yields when $\sigma_1 = 2k$ , point B in Figure 18. Prior to yield the behavior of the material is governed by Equation 479a

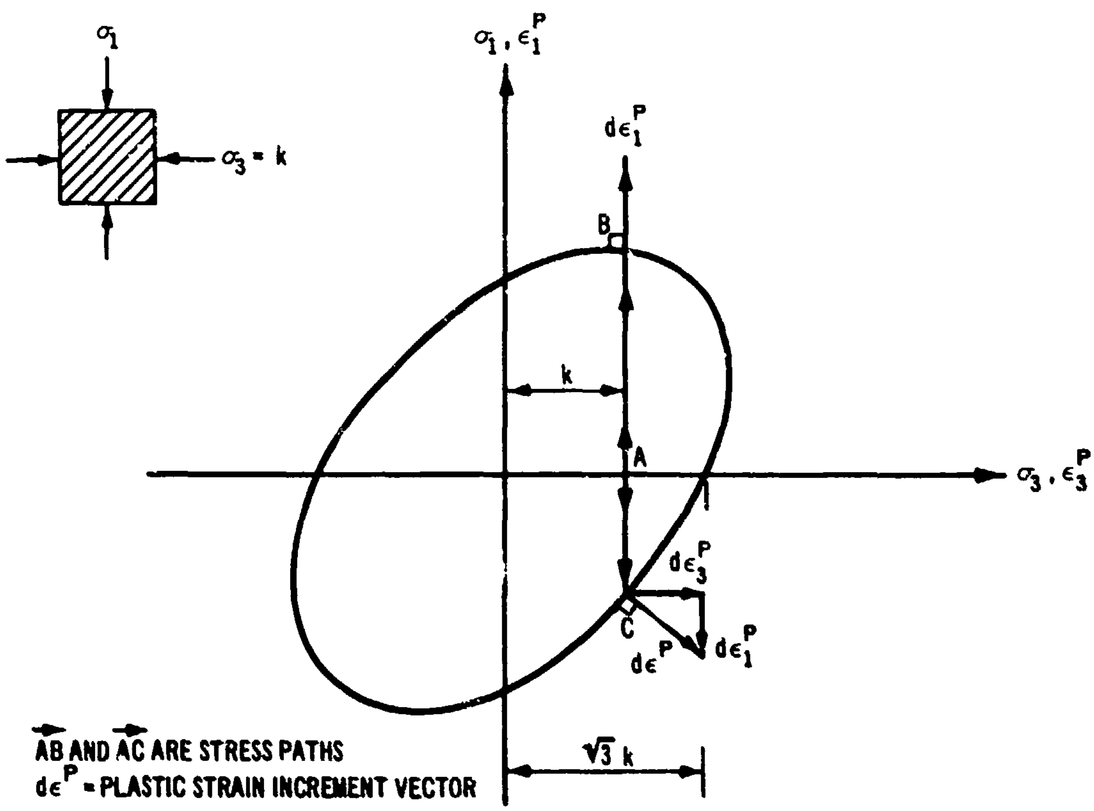



Figure 18. Von Mises yield curve for special plane stress condition

$$d\sigma_1 = \frac{9KG}{3K + G} d\varepsilon_1 = E\, d\varepsilon_1 \tag{497a}$$

$$d\varepsilon_3 = -\left(\frac{3K - 2G}{6K + 2G}\right) d\varepsilon_1 = -\nu\, d\varepsilon_1 \tag{497b}$$

$$d\varepsilon_2 = d\varepsilon_3 \tag{497c}$$

At point B the material yields and it follows from Equation 459 that

$$d\varepsilon_3^p = 0 \tag{498a}$$

$$d\varepsilon_2^p = -d\varepsilon_1^p \tag{498b}$$

Unlimited plastic deformation takes place at yield. It is noted from Equation 498 that, as expected, $d\varepsilon_{kk}^p = 0$ . If we now repeat the same test and change the direction of $\sigma_1$ (i.e., a tension test), we find that the material yields when $\sigma_1 = -k$ , point C in Figure 18. At point C the material yields in tension and from Equation 459 it follows that

$$d\varepsilon_2^p = 0 \tag{499a}$$

$$d\varepsilon_1^p = -d\varepsilon_3^p \tag{499b}$$

The concept of normality can be demonstrated from this simple example by superimposing the plastic strain coordinates on the stress coordinates in Figure 18. As shown in Figure 18, in the case of the compression test $d\varepsilon_3^p = 0$ and the plastic strain increment vector $d\varepsilon_1^p$ is perpendicular to the yield surface at point B. In the case of the tension test, on the other hand, $d\varepsilon_1^p = -d\varepsilon_3^p$ indicating that the plastic strain increment vector is perpendicular to the yield surface at point C.

<u>Drucker-Prager material</u>

138. The Von Mises yield condition was modified by Drucker and Prager[12] to include the effects of the hydrostatic stress on the shearing resistance of the material. The yield function $\mathcal{f}$ was assumed to take the following form

$$\mathcal{f} = \sqrt{J_2'} - \alpha_f J_1 = k \tag{500}$$

where $\alpha_f$ , a positive material constant, represents the frictional strength of the material. Equation 500 describes a right circular cone in the principal stress space (Figure 19). Substituting Equation 500

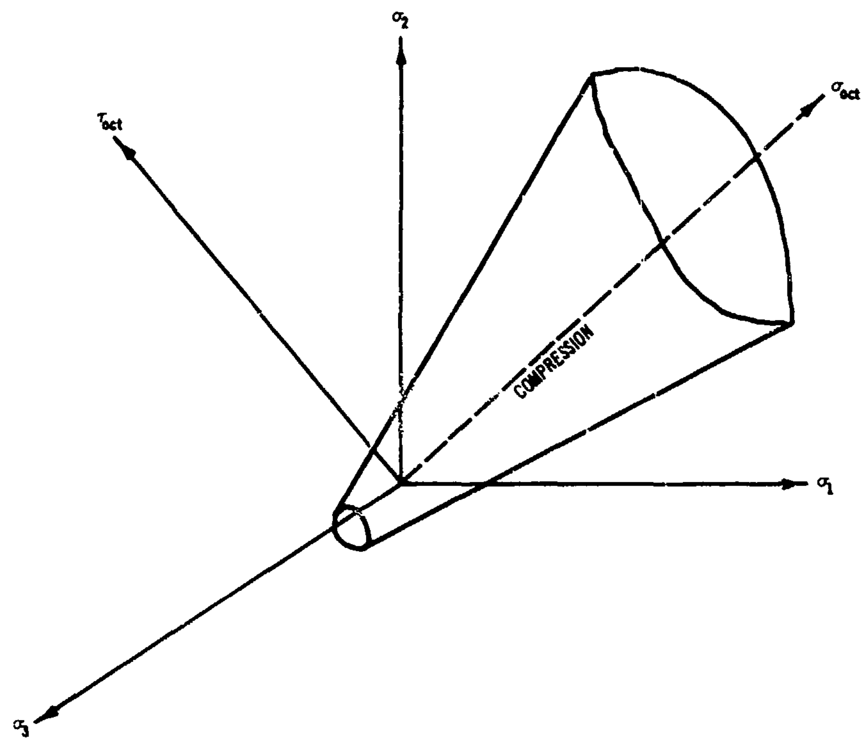



Figure 19. Drucker-Prager yield surface in principal stress space

into Equation 469 we obtain the following stress-strain relationship associated with the Drucker-Prager yield function

$$d\epsilon_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K}\delta_{ij} + \left( \frac{\frac{G}{\sqrt{J_2'}} S_{mn}\, dE_{mn} - 3K\alpha_f\, dI_1}{9K\alpha_f^2 + G} \right) \times \left( \frac{S_{ij}}{2\sqrt{J_2'}} - \alpha_f \delta_{ij} \right) \qquad (501)$$

From Equation 501 it follows that

$$d\varepsilon_{kk}^{p} = -3\alpha_f \left( \frac{\frac{G}{\sqrt{J_2'}} S_{mn}\, dE_{mn} - 3K\alpha_f\, dI_1}{9K\alpha_f^2 + G} \right) \tag{502}$$

indicating that for Drucker-Prager material, as a consequence of dependency of yield function on hydrostatic stress, plastic deformation is accompanied by volume expansion (it is noted from Equation 502 that if $\alpha_f = 0$ the plastic volumetric strain is zero also). The increment of total volumetric strain $dI_1$ can be determined from Equations 501 and 500. From Equation 501 we have

$$dI_1 = \frac{dJ_1}{3K} - 3\alpha_f \left[ \frac{\frac{G}{\sqrt{J_2'}} \left( \sigma_{mn}\, d\varepsilon_{mn} - J_1\, dI_1/3 \right) - 3K\alpha_f\, dI_1}{9K\alpha_f^2 + G} \right] \tag{503}$$

Solving for $dI_1$ and considering the fact that during plastic deformation $\sqrt{J_2'} - \alpha_f J_1 = k$ (Equation 500), we obtain

$$dI_1 = \frac{\sqrt{J_2'}\, dJ_1}{3KGk} \left( 9K\alpha_f^2 + G \right) - \frac{3\alpha_f}{k} \sigma_{mn}\, d\varepsilon_{mn} \tag{504}$$

The increment of plastic volumetric strain $d\varepsilon_{kk}^{p}$ then becomes

$$d\varepsilon_{kk}^{p} = \frac{\sqrt{J_2'}\, dJ_1}{3KGk} \left( 9K\alpha_f^2 + G \right) - \frac{3\alpha_f}{k} \sigma_{mn}\, d\varepsilon_{mn} - \frac{dJ_1}{3K} \tag{505}$$

It should be pointed out that the volume change is due to scalar nonlinearity and represents uniform dilatation. For example, consider a simple shearing stress defined by the following stress increment tensor

$$d\sigma_{ij} = \begin{bmatrix} 0 & d\sigma_{12} & 0 \\ d\sigma_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{506}$$

From Equations 501 and 504 it follows that for this state of stress

$$d\varepsilon_1 = d\varepsilon_2 = d\varepsilon_3 \tag{507}$$

That is, there are no normal deviatoric strains associated with the volume expansion.

139. Substituting Equation 500 in Equation 470 we obtain the following relationship for the stress increment tensor for the Drucker-Prager material

$$d\sigma_{ij} = 2G\ dE_{ij} + K\ dI_1\ \delta_{ij} - \left(\frac{\frac{G}{\sqrt{J_2'}} S_{mn}\ dE_{mn} - 3K\alpha_f\ dI_1}{9K\alpha_f^2 + G}\right) \times \left(\frac{G}{\sqrt{J_2'}} S_{ij} - 3K\alpha_f \delta_{ij}\right) \tag{508}$$

Equation 508 (or Equation 501) governs the behavior of Drucker-Prager material. The effect of the dependency of the yield function on hydrostatic stress can be further demonstrated by examining the behavior of Drucker-Prager material under uniaxial state of strain (Equation 480). The elastic behavior of the material is given by Equation 481. The material yields when

$$\frac{1}{\sqrt{3}} (\sigma_1 - \sigma_2) - \alpha_f(\sigma_1 + 2\sigma_2) = k \tag{509}$$

In view of Equations 481 and 509, the value of vertical stress $\sigma_1$ at yield becomes

$$\sigma_1 = \frac{\sqrt{3}\ (3K + 4G)}{6G - 9\sqrt{3}\ K\alpha_f} k = \frac{\sqrt{3}\ Mk}{2G - 3\sqrt{3}\ K\alpha_f} \tag{510}$$

It is noted that if $\alpha_f$ is set to zero, Equation 510 reduces to Equation 484, the corresponding expression for Prandtl-Reuss material. The effect of $\alpha_f$ in this case is to increase the value of the vertical stress $\sigma_1$ at yield. When $\sigma_1$ reaches the value given by Equation 510, the material yields. Continued application of vertical stress causes the material to move along the yield surface, undergoing both elastic and plastic deformation. From Equation 508 the incremental relation between vertical stress and vertical strain becomes

$$d\sigma_1 = \left(K + \frac{4}{3} G\right) d\varepsilon_1 - \frac{\left(\frac{2\sqrt{3}}{3} G - 3K\alpha_f\right)^2}{9K\alpha_f^2 + G} d\varepsilon_1 \tag{511}$$

Again it is noted that when $\alpha_f$ is set to zero Equation 511 reduces to Equation 492, the corresponding expression for Prandtl-Reuss material. As was pointed out previously, for Drucker-Prager material plastic deformation is accompanied by volume expansion (see Equation 502). Accordingly, using Equations 504 and 509, we obtain the following incremental relation for volumetric strain in the case of uniaxial strain test

$$dJ_1 = \frac{9K\alpha_f \left(\frac{2\sqrt{3}}{3} G - 3K\alpha_f\right)}{9K\alpha_f^2 + G} dI_1 + 3K\, dI_1 \tag{512}$$

When $\alpha_f$ is set to zero, Equation 512 reduces to the corresponding expression for elastic material. The increment of plastic volumetric strain then becomes

$$d\varepsilon^p_{kk} = \frac{\alpha_f(9K\alpha_f - 2\sqrt{3}\, G)}{3KG(1 + 2\sqrt{3}\, \alpha_f)} \cdot dJ_1 \tag{513}$$

In order for the uniaxial strain-stress path to reach the yield surface, the following condition should hold

$$\frac{2G}{\sqrt{3}\, K} > 3\alpha_f \tag{514}$$

Therefore, as expected, the increment of plastic volumetric strain is negative (expansion).

## Work-Hardening Plastic Material

140. In the case of work-hardening plastic material, the yield surface $f$ is not fixed but expands, or translates, as plastic deformation takes place. The material can then sustain stresses beyond those required to reach the initial yield condition. Therefore, we can use a loading concept in the case of work-hardening plastic material according to the direction of the stress increment tensor $d\sigma_{ij}$ (viewed as vector).

During loading from a point on a given yield surface the stress vector is pointing outward and thus $(\partial f/\partial \sigma_{ij})d\sigma_{ij} > 0$ . During unloading the stress vector is pointing inward and thus $(\partial f/\partial \sigma_{ij})d\sigma_{ij} < 0$ . Accordingly for work-hardening plastic material we define

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \quad \text{loading} \tag{515a}$$

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \quad \text{unloading} \tag{515b}$$

The condition $(\partial f/\partial \sigma_{ij})d\sigma_{ij} = 0$ (i.e., when $d\sigma_{ij}$ is tangent to yield surface) is known as neutral loading and produces no plastic deformation in the case of work-hardening material. The stability condition for work-hardening plastic material is given as

$$d\sigma_{ij}\, d\varepsilon_{ij} > 0 \tag{516a}$$

$$d\sigma_{ij}\, d\varepsilon^{p}_{ij} \geq 0 \tag{516b}$$

where, unlike the ideal plastic material, the equality sign in Equation 516b holds only when $d\varepsilon^{p}_{ij} = 0$ . For work-hardening plastic material Drucker[11] has shown that the expression for plastic strain increment tensor is similar to Equation 459 where the proportionality factor $\tilde{\Lambda}$ depends on stress, plastic deformation, and history of plastic deformation. We can, therefore, use Equation 460, in conjunction with the loading conditions given by Equation 515, for calculating the strain increment tensor. During loading from a point on the yield surface $\left((\partial f/\partial \sigma_{ij})d\sigma_{ij} > 0\right)$, Equation 460 governs the behavior of the material. In the elastic range, and during unloading from a point on the yield surface $\left((\partial f/\partial \sigma_{ij})d\sigma_{ij} < 0\right)$, the behavior of the material is governed by Equation 453. When $(\partial f/\partial \sigma_{ij})d\sigma_{ij} = 0$ (neutral loading), $d\varepsilon^{p}_{ij} = 0$ and Equations 460 and 453 become identical (thus establishing continuity at a load-unload interface).

141. We now adopt a yield condition of the following type

$$f = f\left(\sigma_{ij}\ ,\ \varepsilon^{p}_{mn}\right) = k \tag{517}$$

for strain-hardening material. Equation 517 indicates that the yield surface is not fixed in the principal space and that it changes as plastic deformation takes place. We further assume that k is a constant. Following the same procedure as was used to derive an expression for $\tilde{\Lambda}$ in the case of ideal plastic material we obtain

$$\tilde{\Lambda} = \frac{\frac{\partial f}{\partial \sigma_{ij}} d\epsilon_{ij} + \frac{3K - 2G}{6G} dI_1 \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij}}{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \frac{3K - 2G}{6G}\left(\frac{\partial f}{\partial \sigma_{ij}} \delta_{ij}\right)^2 - \frac{\partial f}{\partial \epsilon^p_{ij}} \frac{\partial f}{\partial \sigma_{ij}}} \qquad (518)$$

Equation 518 is the expression for the proportionality factor $\tilde{\Lambda}$ associated with the strain-hardening yield condition given by Equation 517. It is noted that Equation 518 reduces to Equation 464 when the dependency of the yield function on the plastic strain disappears (i.e., ideal plastic material). In view of Equations 518 and 460, the strain increment tensor associated with the yield condition of Equation 517 becomes

$$d\epsilon_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K} \delta_{ij} + \left[\frac{\frac{\partial f}{\partial \sigma_{mn}} d\epsilon_{mn} + \frac{3K - 2G}{6G} dI_1 \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}}{\frac{\partial f}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{mn}} + \frac{3K - 2G}{6G}\left(\frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}\right)^2 - \frac{\partial f}{\partial \epsilon^p_{mn}} \frac{\partial f}{\partial \sigma_{mn}}}\right] \frac{\partial f}{\partial \sigma_{ij}} \qquad (519)$$

We can also derive an expression for the stress increment tensor

$$d\sigma_{ij} = 2G\, dE_{ij} + K\, dI_1\, \delta_{ij} - \left[\frac{\frac{\partial f}{\partial \sigma_{mn}} d\epsilon_{mn} + \frac{3K - 2G}{6G} dI_1 \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}}{\frac{\partial f}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{mn}} + \frac{3K - 2G}{6G}\left(\frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}\right)^2 - \frac{\partial f}{\partial \epsilon^p_{mn}} \frac{\partial f}{\partial \sigma_{mn}}}\right] \times \left[\left(\frac{3K - 2G}{3} \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}\right) \delta_{ij} + 2G \frac{\partial f}{\partial \sigma_{ij}}\right] \qquad (520)$$

For a number of engineering materials, soils in particular, the yield function $f$ is expressed in terms of $J_1$, $\sqrt{J_2'}$, and $\varepsilon_{kk}^p$, i.e.,

$$f\left(\sigma_{ij}, \varepsilon_{mn}^p\right) = f\left(J_1, \sqrt{J_2'}, \varepsilon_{kk}^p\right) \tag{521}$$

For the above specification of $f$, Equations 519 and 520 become

$$d\varepsilon_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K}\delta_{ij} + \left[\frac{3K\, dI_1 \dfrac{\partial f}{\partial J_1} + \dfrac{G}{\sqrt{J_2'}} \dfrac{\partial f}{\partial \sqrt{J_2'}} S_{mn}\, dE_{mn}}{9K\left(\dfrac{\partial f}{\partial J_1}\right)^2 + G\left(\dfrac{\partial f}{\partial \sqrt{J_2'}}\right)^2 - 3\dfrac{\partial f}{\partial \varepsilon_{kk}^p}\dfrac{\partial f}{\partial J_1}}\right] \times \left(\frac{\partial f}{\partial J_1}\delta_{ij} + \frac{1}{2\sqrt{J_2'}}\frac{\partial f}{\partial \sqrt{J_2'}} S_{ij}\right) \tag{522}$$

and

$$d\sigma_{ij} = 2G\, dE_{ij} + K\, dI_1\, \delta_{ij} - \left[\frac{3K\, dI_1 \dfrac{\partial f}{\partial J_1} + \dfrac{G}{\sqrt{J_2'}} \dfrac{\partial f}{\partial \sqrt{J_2'}} S_{mn}\, dE_{mn}}{9K\left(\dfrac{\partial f}{\partial J_1}\right)^2 + G\left(\dfrac{\partial f}{\partial \sqrt{J_2'}}\right)^2 - 3\dfrac{\partial f}{\partial \varepsilon_{kk}^p}\dfrac{\partial f}{\partial J_1}}\right] \times \left(3K\frac{\partial f}{\partial J_1}\delta_{ij} + \frac{G}{\sqrt{J_2'}}\frac{\partial f}{\partial \sqrt{J_2'}} S_{ij}\right) \tag{523}$$

It is noted that Equations 522 and 523 reduce to Equations 469 and 470, respectively, when the dependency of the yield function on $\varepsilon_{kk}^p$ disappears.

142. In order to demonstrate the application of Equation 522 (or Equation 523) let us consider an elliptic yield function defined by the following equation (Figure 20)

$$f\left(J_1, \sqrt{J_2'}, Y\right) = J_1\left(J_1 - Y\right) + \left(\frac{\sqrt{J_2'}}{Q}\right)^2 = 0 \tag{524}$$

Equation 524 has been used successfully for modeling the stress-strain behavior of earth materials.[13] For a first-order approximation, the variable $Y$, which controls the expansion of the yield surface, is assumed to take the form

$$Y = \bar{A}\varepsilon_{kk}^p \tag{525}$$

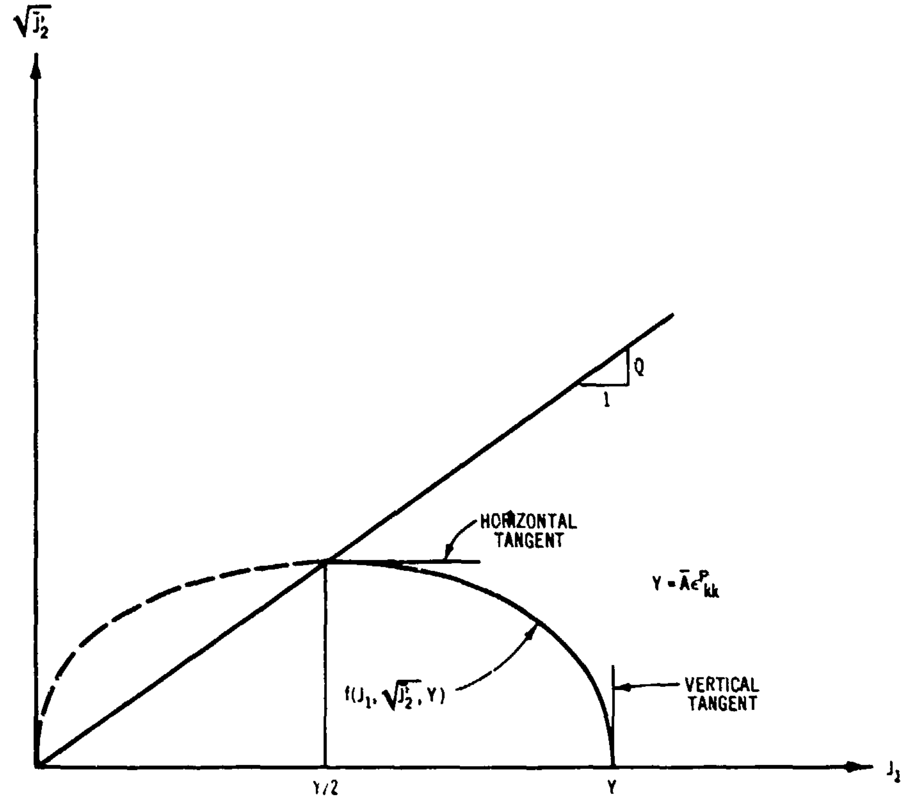



Figure 20. Work-hardening elliptic yield surface

where $\bar{A}$ is a material constant which must be determined experimentally. In order to obtain the constitutive equation for the assumed work-hardening yield surface we substitute Equation 524, for the yield surface $f$, into Equation 522. Completing the substitution and considering the fact that

$$\frac{\partial f}{\partial \epsilon^p_{kk}} = \frac{\partial f}{\partial Y} \frac{\partial Y}{\partial \epsilon^p_{kk}} \tag{526}$$

we obtain

$$d\epsilon_{ij} = \frac{dS_{ij}}{2G} + \frac{dJ_1}{9K}\delta_{ij} + \left[\frac{3KQ^2\left(2J_1 - Y\right) dI_1 + 2GS_{mn}\, dE_{mn}}{9KQ^4\left(2J_1 - Y\right)^2 + 4G\bar{J}_2' + 3Q^4\bar{A}J_1\left(2J_1 - Y\right)}\right] \times \left[Q^2\left(2J_1 - Y\right)\delta_{ij} + S_{ij}\right] \tag{527}$$

143. Let us now examine the behavior of Equation 527 under hydrostatic state of stress (Figure 6a). For hydrostatic state of stress $J_1 = Y$ and Equation 527 becomes

$$dI_1 = \frac{(3K + \bar{A})}{3K\bar{A}} dJ_1 = \frac{1}{K_e} dJ_1 \tag{528}$$

It should be noted that for this state of stress the same results could have been obtained directly from Equations 452, 453, and 525 without recourse to Equation 527. For virgin loading, Equation 528 can be integrated to yield (assuming zero initial pressure and volumetric strain)

$$J_1 = K_e I_1 \tag{529}$$

During purely elastic deformation (Equation 453)

$$dJ_1 = 3K\, dI_1 \tag{530}$$

Since $K_e < 3K$ (Equation 528), it follows that plastic compaction produces an apparent softening of the effective bulk modulus. Figure 21 depicts the behavior of the material under hydrostatic state of stress. The behavior of the material from point 1 to point 2 is governed by Equation 529 (the material undergoes plastic as well as elastic deformation). If the material is unloaded from point 2 to point 3, and then reloaded from point 3 to point 2, the behavior is elastic and the response of the material is governed by Equation 530.

144. Let us next examine the behavior of Equation 527 under a constant-pressure shear test (Figure 6c). The qualitative behavior of the model is depicted in Figure 22. The material is first hydrostatically loaded from point 1 to point 2. The response of the material from point 1 to point 2 is governed by Equation 529 and is identical to that shown in Figure 21 (the material undergoes both plastic and elastic deformation). The material is then sheared from point 2 to point 3 by increasing $\sqrt{J_2'}$ while $J_1$ is kept constant. Since $J_1$ is kept constant, all volume changes from point 2 to point 3 are plastic. From

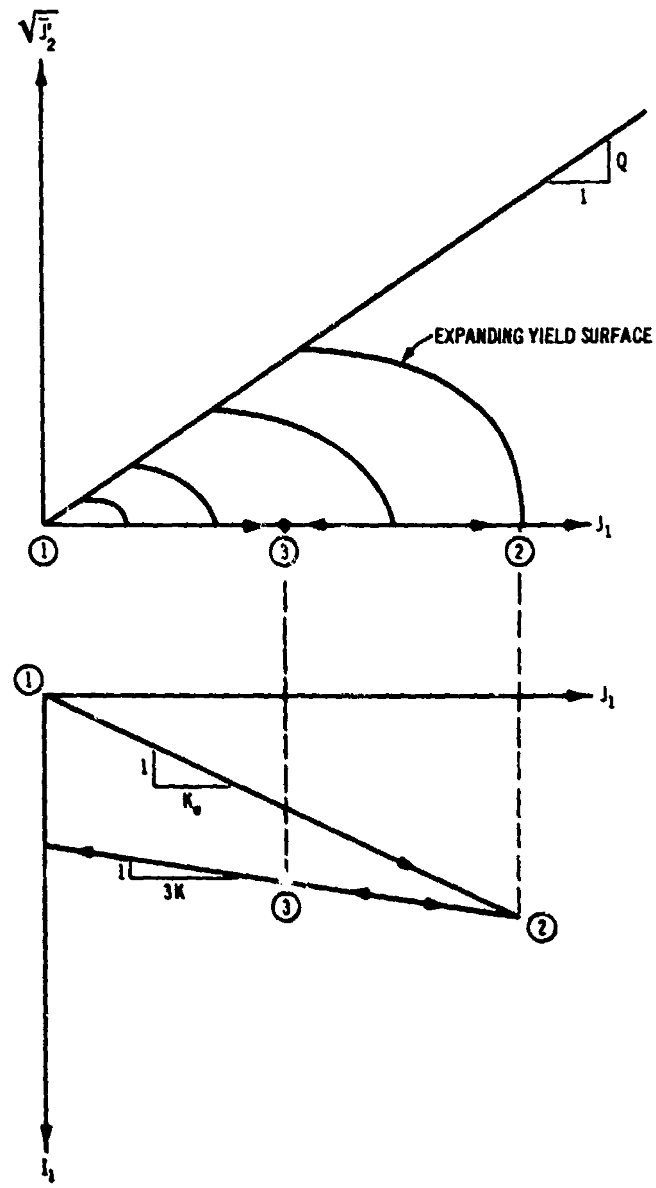



Figure 21. Behavior of work-hardening elastic-plastic material under hydrostatic state of stress

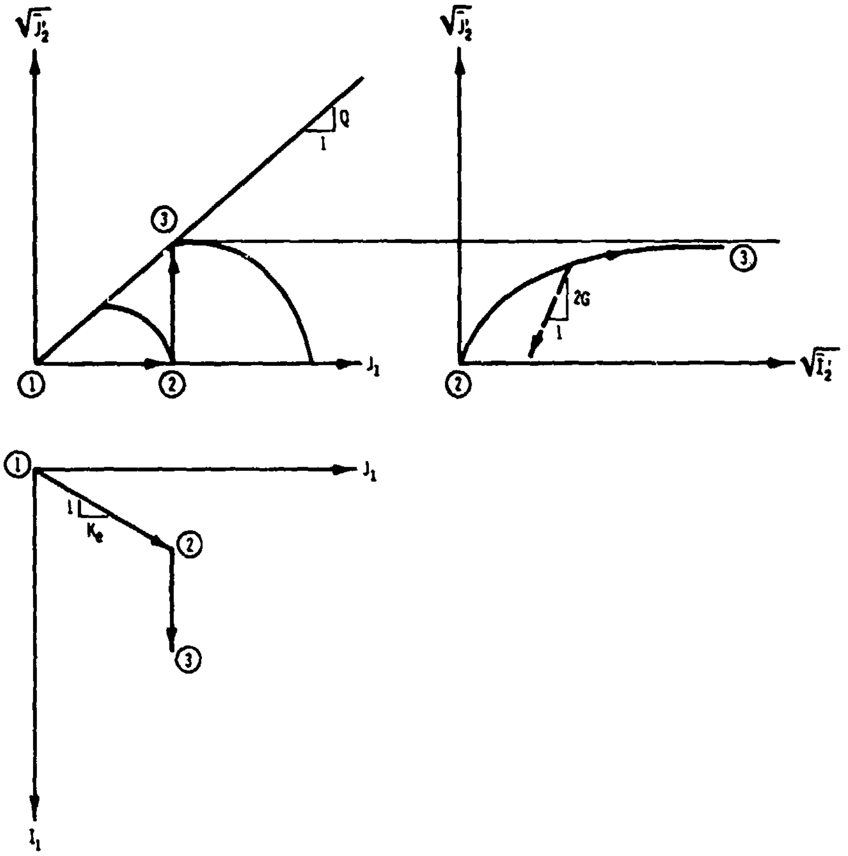



Figure 22 Behavior of work-hardening elastic-plastic material during constant-pressure shear test

Equation 459 it follows that the increment of plastic volumetric strain is given as

$$d\varepsilon^p_{kk} = 3\tilde{\Lambda} \frac{\partial f}{\partial J_1} \tag{531}$$

In view of Equation 524, Equation 531 becomes

$$d\varepsilon^p_{kk} = 3\tilde{\Lambda}(2J_1 - Y) \tag{532}$$

Since $\tilde{\Lambda}$ is positive, Equation 532 indicates that the plastic volumetric strain during the shearing process is positive (compaction). At point 3 $d\varepsilon^p_{kk} = 0$ (normality condition) and the yield surface ceases to expand. The shearing response of material, expressed in terms of $\sqrt{J_2'}$ versus $\sqrt{I_2'}$, then reaches its maximum value (for the particular value

of $J_1$ at point 2) asymptotically at point 3. As shown by the dashed lines in Figure 22, if the material were to unload from any point during the shearing process it will behave as a linear elastic material. This simple example points out the basic difference between ideal and work-hardening plastic materials. That is, for ideal plastic materials the yield surface is fixed and does not expand during plastic deformation. Unlimited plastic flow takes place at the onset of yielding. In the case of work-hardening material, on the other hand, the yield surface moves, or expands, causing the material to harden as plastic deformation takes place.

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# APPENDIX A: SELECTED BIBLIOGRAPHY

This appendix contains 78 references that are related to subject matter presented in the main text of this report. These references are in addition to those listed following the main text and are given in alphabetical order. For easy reference and informational purposes, the references in this appendix are cataloged in Table A1 in accordance with their relation to three basic material models: incremental models, plasticity, and viscoelasticity.

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In accordance with letter from DAEN-RDC, DAEN-ASI dated 22 July 1977, Subject: Facsimile Catalog Cards for Laboratory Technical Publications, a facsimile catalog card in Library of Congress MARC format is reproduced below.


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